

Logarithmic nonabelian Hodge theory in characteristic p

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ABSTRACT

Given a morphism $X \rightarrow S$ of log schemes of characteristic $p > 0$ and a lifting \mathcal{X}/\mathcal{S} of X' over S modulo p^2 , we use Lorenzon's indexed algebras $\mathcal{A}_X^{\text{gp}}$ and $\mathcal{B}_{X/S}$ to construct an equivalence between \mathcal{O}_X -modules with nilpotent integrable connection and indexed $\mathcal{B}_{X/S}$ -modules with nilpotent $\mathcal{B}_{X/S}$ -linear Higgs field. If either satisfies a stricter nilpotence condition, we find an isomorphism between the de Rham cohomology of the connection and the Higgs cohomology of the Higgs field.

1. Introduction

The classical Riemann-Hilbert correspondence gives an equivalence between the category of coherent modules E with integrable connection ∇ on a complex manifold X and the category of locally constant sheaves of complex vector spaces V on X . Moreover, there is a natural isomorphism

$$H^n(X, V) \simeq H^n(X, E \otimes \Omega_{X/\mathbb{C}}),$$

where the right hand side is the hypercohomology of the de Rham complex of ∇ . However, the maps in this complex are only \mathbb{C} -linear in general. On the other hand, if V is constant and X is the analytic space associated to a projective scheme over \mathbb{C} , Hodge theory tells us that in fact,

$$H^n(X, V) \simeq \bigoplus_{i+j=n} H^i(X, E \otimes \Omega_{X/\mathbb{C}}^j).$$

In general, if V is not constant, this last statement is no longer true. Simpson [Sim92] resolved this by defining a partial equivalence between modules with integrable connection (E, ∇) and modules with Higgs field (E', θ') (with a stability condition on both sides), such that the de Rham complex of (E, ∇) and the Higgs complex of (E', θ') have the same hypercohomology. Recall that a *Higgs field* on E' is a map $\theta' : E' \rightarrow E' \otimes_{\mathcal{O}_X} \Omega_{X/\mathbb{C}}^1$ which, instead of satisfying the Leibniz rule, is simply \mathcal{O}_X -linear and satisfies the integrability condition $\theta' \wedge \theta' = 0$, so that θ' induces a complex of \mathcal{O}_X -modules

$$E' \rightarrow E' \otimes_{\mathcal{O}_X} \Omega_{X/\mathbb{C}}^1 \rightarrow E' \otimes_{\mathcal{O}_X} \Omega_{X/\mathbb{C}}^2 \rightarrow \cdots.$$

(This is equivalent to an extension of the \mathcal{O}_X -module structure on E' to an S - $\mathcal{T}_{X/\mathbb{C}}$ -module structure.)

On schemes in characteristic p , the straightforward generalization of the Riemann-Hilbert correspondence requires that the connection ∇ have vanishing p -curvature. The construction of the p -curvature is as follows: recall that given a morphism $X \rightarrow S$ of schemes of characteristic p , for a derivation $D : \mathcal{O}_X \rightarrow \mathcal{O}_X$ over \mathcal{O}_S , its p th iterate $D^{(p)}$ is again a derivation. Now given an \mathcal{O}_X -module E with integrable connection ∇ , it turns out that

$$D \mapsto \psi_D := (\nabla_D)^p - \nabla_{D^{(p)}} \in \mathcal{E}nd_{\mathcal{O}_S}(E)$$

induces a Frobenius-linear map $\mathcal{T}_{X/S} \rightarrow F_{X/S*} \mathcal{E}nd_{\mathcal{O}_X}(E)$, or equivalently an \mathcal{O}_X -linear map $\psi : E \rightarrow E \otimes_{\mathcal{O}_X} F_{X/S}^* \Omega_{X'/S}^1$. (The appendix to this paper gives a new proof of this fact which is more conceptual than previous proofs, for example that in [Kat70].) The classical Cartier descent theorem [Kat70] then gives an equivalence between the category of modules with integrable connection with vanishing p -curvature and the category of $\mathcal{O}_{X'}$ -modules.

However, many connections of interest do not have vanishing p -curvature. In their recent work [OV], Ogus and Vologodsky define a more general equivalence between modules with integrable connection (E, ∇) whose p -curvature is nilpotent of order less than p and modules with Higgs field (E', θ') with θ' nilpotent of order less than p . This equivalence depends on a lifting $\tilde{X}' \rightarrow \tilde{S}$ modulo p^2 of X' over S , where X' is the target of the relative Frobenius map $F_{X/S} : X \rightarrow X'$. Furthermore, they found that if the p -curvature of ∇ is nilpotent of sufficiently low order, then the de Rham complex of (E, ∇) and the Higgs complex of (E', θ') are naturally isomorphic in the appropriate derived category, so that they again have the same hypercohomology.

Log geometry was created to deal with problems in compactification and singularities; thus, connections on log schemes provide a language for studying differential equations with log poles. This is an important case to study since many natural connections do have log poles. A log scheme is a scheme X with a sheaf of commutative monoids \mathcal{M}_X and a map $\alpha : \mathcal{M}_X \rightarrow \mathcal{O}_X^\times$, where \mathcal{O}_X^\times is the multiplicative monoid of \mathcal{O}_X , such that α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \rightarrow \mathcal{O}_X^*$. For example, given a divisor with normal crossings D , we may define $\mathcal{M}_X := \mathcal{O}_X \cap i_* \mathcal{O}_Y^*$, where $Y := X \setminus D$ and

$i : Y \hookrightarrow X$ is the open immersion, and we define $\alpha : \mathcal{M}_X \rightarrow \mathcal{O}_X$ to be the natural inclusion. Then for a morphism $X \rightarrow S$ of log schemes, Kato defines a sheaf of relative logarithmic differentials $\Omega_{X/S}^1$ [Kat88], which allows the natural extension of the notion of modules with integrable connection; in the case above of a divisor with normal crossings, this is just the classical sheaf $\Omega_{X/S}^1(\log D)$ of differentials with log poles along D .

However, on log schemes of characteristic p , in addition to the p -curvature there is another obstruction, called the residue, to the classical Riemann-Hilbert correspondence; hence the straightforward generalization of the Riemann-Hilbert correspondence requires that the connection ∇ have vanishing residue in addition to vanishing p -curvature. Lorenzon [Lor00] corrected this by introducing an $\overline{\mathcal{M}}_X^{\text{gp}}$ -indexed ring $\mathcal{A}_X^{\text{gp}}$ with canonical connection d , such that defining $\mathcal{B}_{X/S} := (\mathcal{A}_X^{\text{gp}})^{d=0}$, one gets an equivalence between indexed $\mathcal{A}_X^{\text{gp}}$ -modules with integrable connection compatible with d whose p -curvature vanishes and indexed $\mathcal{B}_{X/S}$ -modules. This work was inspired by Tsuji's work generalizing the fundamental exact sequence

$$0 \rightarrow \mathcal{O}_{X'}^* \rightarrow \mathcal{O}_X^* \xrightarrow{d \log} Z_{X/S}^1 \xrightarrow{\pi^* - C} \Omega_{X'/S}^1 \rightarrow 0$$

to the case of log schemes, replacing \mathcal{O}_X^* by $\mathcal{M}_X^{\text{gp}}$ [Tsu96].

The aim of this paper is to extend the theory of Ogus and Vologodsky in [OV] to the case of log schemes. The main result is that given a lifting \mathcal{X}/S of X' over S modulo p^2 , we get an equivalence $C_{\mathcal{X}/S}$ between the category of \mathcal{O}_X -modules with integrable connection (E, ∇) whose p -curvature is nilpotent of level less than p and the category of $\overline{\mathcal{M}}_X^{\text{gp}}$ -indexed $\mathcal{B}_{X/S}$ -modules with $\mathcal{B}_{X/S}$ -linear Higgs field (E', θ') which is nilpotent of level less than p . (Here $\overline{\mathcal{M}}_X^{\text{gp}}$ denotes the quotient $\mathcal{M}_X^{\text{gp}}/\mathcal{O}_X^*$.) Furthermore, this specializes to give an equivalence between \mathcal{O}_X -modules with connection whose p -curvature and residue are both nilpotent and $\mathcal{O}_{X'}$ -modules with nilpotent Higgs field. In both cases, the de Rham complex of (E, ∇) and the Higgs complex of (E', θ') are once again isomorphic in the derived category.

We give here a brief sketch of the construction of $C_{\mathcal{X}/S}$ and its pseudoinverse $C_{\mathcal{X}/S}^{-1}$, assuming for simplicity we have a lifting $\tilde{X} \rightarrow \tilde{S}$ in addition to the lifting of X' . Then the sheaf of liftings of $F_{X/S} : X \rightarrow X'$ to a map $\tilde{X} \rightarrow \tilde{X}'$ is a torsor over $F_{X/S}^* \Omega_{X'/S}^1$. This sheaf is representable by an affine scheme $\mathcal{L}_{\mathcal{X}/S} = \mathbf{Spec} \mathcal{K}_{\mathcal{X}/S}$ over X ; in addition, the sheaf extends naturally to the crystalline site $\text{Crys}(X/S)$, which induces a natural connection on $\mathcal{K}_{\mathcal{X}/S}$. Defining $\mathcal{K}_{\mathcal{X}/S}^A := \mathcal{K}_{\mathcal{X}/S} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}$, we then have

$$C_{\mathcal{X}/S}(E) \simeq (E \otimes_{\mathcal{O}_X} \mathcal{K}_{\mathcal{X}/S}^A)^{\nabla_{\text{tot}}}$$

with Higgs field induced by the p -curvature of $\mathcal{K}_{\mathcal{X}/S}$, and

$$C_{\mathcal{X}/S}^{-1}(E') \simeq (E' \otimes_{\mathcal{B}_{X/S}} \mathcal{K}_{\mathcal{X}/S}^A)_0^{\theta_{\text{tot}}}$$

with connection acting on $\mathcal{K}_{\mathcal{X}/S}$.

A key technical result which is central to the proofs in [OV] is the fact that for a scheme X of characteristic p , the ring of PD differential operators $\mathcal{D}_{X/S}$ is an Azumaya algebra over its center, which is isomorphic to $S \cdot \mathcal{T}_{X'/S}$ via the map $D \mapsto D^p - D^{(p)}$. (This means that locally, after a flat base extension, it is isomorphic to a matrix algebra.) This is no longer true in general in the case of log schemes; however, what we find is that if we define $\tilde{\mathcal{D}}_{X/S} := \mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}$ with the appropriate multiplication, then $\tilde{\mathcal{D}}_{X/S}$ is an indexed Azumaya algebra over its center, which is isomorphic to $\mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} S \cdot \mathcal{T}_{X'/S}$. The first section is mainly devoted to an elaboration and proof of this result. The first subsection discusses the theory of indexed Azumaya algebras; perhaps one interesting point which should be mentioned here is the fact that we need to define $\mathcal{H}om$ in the indexed case to allow morphisms which shift degree. The second subsection reviews the construction of $\mathcal{A}_X^{\text{gp}}$ and

the connection d and explores the structure of $\mathcal{B}_{X/S}$. The third subsection then defines $\tilde{\mathcal{D}}_{X/S}$ and proves that it is an indexed Azumaya algebra over its center.

The second section begins with a construction of the scheme representing a general torsor over a locally free sheaf. We then define the $F_{X/S}^* \Omega_{X'/S}^1$ -torsor of liftings of Frobenius $\mathcal{L}_{X/S}$ and its extension to the crystalline site, which gives us a corresponding crystal of $\mathcal{O}_{X/S}$ -algebras $\mathcal{K}_{X/S}$. We proceed to calculate the corresponding connection and its p -curvature explicitly (the latter turns out to be simply the map $d : \mathcal{O}_{\mathcal{L}} \rightarrow \Omega_{\mathcal{L}/X}^1 \simeq \mathcal{O}_{\mathcal{L}} \otimes_{\mathcal{O}_X} F_{X/S}^* \Omega_{X'/S}^1$, where by abuse of notation we also let \mathcal{L} be the scheme representing the torsor $\mathcal{L}_{X/S}$) and to discuss a functorality property of this construction.

The third section gives a construction of the Cartier transform $C_{X/S}$ described above; the key observation is that defining

$$\check{\mathcal{K}}_{X/S}^A := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{K}_{X/S}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}},$$

then $\check{\mathcal{K}}_{X/S}^A$ is a splitting module for $\tilde{\mathcal{D}}_{X/S}^\gamma := \tilde{\mathcal{D}}_{X/S} \otimes_{S \cdot \mathcal{T}_{X'/S}} \hat{\Gamma} \cdot \mathcal{T}_{X'/S}$. We thus get an equivalence between the slightly larger categories of $\tilde{\mathcal{D}}_{X/S}^\gamma$ -modules and $\hat{\Gamma} \cdot \mathcal{T}_{X'/S} \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S}$ -modules, which have the advantage of being \otimes -categories. We then prove that after a sign change the equivalence given by Azumaya algebra theory gives the formulas above in the case of quasiniptent modules. We also give a local version of the transform which works more generally for $\tilde{\mathcal{D}}_{X/S} \otimes_{S \cdot \mathcal{T}_{X'/S}} \hat{S} \cdot \mathcal{T}_{X'/S}$ -modules, but depends on a lifting \tilde{F} of $F_{X/S}$, which rarely exists globally. Finally, in the last section, we prove the analogue of Simpson's formality theorem, which states that if the p -curvature of ∇ is nilpotent of level less than $p - d$, where d is the relative dimension of X/S , then the de Rham complex of (E, ∇) is isomorphic in the derived category of complexes of $\mathcal{O}_{X'}$ -modules to the Higgs complex of its Cartier transform.

In their paper Ogus and Vologodsky proved a theorem of Barannikov and Kontsevich, which states that if X is a quasi-projective smooth scheme over \mathbb{C} and $f \in \Gamma(X, \mathcal{O}_X)$ induces a proper map $X \rightarrow \mathbb{A}^1$, then the hypercohomologies of the complexes

$$\mathcal{O}_X \xrightarrow{d+\wedge df} \Omega_{X/\mathbb{C}}^1 \xrightarrow{d+\wedge df} \Omega_{X/\mathbb{C}}^2 \cdots$$

and

$$\mathcal{O}_X \xrightarrow{\wedge df} \Omega_{X/\mathbb{C}}^1 \xrightarrow{\wedge df} \Omega_{X/\mathbb{C}}^2 \cdots$$

have the same finite dimension in every degree. For further research, it would be interesting to see whether by using log geometry one can relax the condition on f .

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2. Indexed Azumaya Algebras

Let X be a topological space, and \mathcal{I} a sheaf of abelian groups on X . Then an \mathcal{I} -indexed ring on X is a sheaf \mathcal{A} along with a degree map $\deg : \mathcal{A} \rightarrow \mathcal{I}$, a zero section $\mathcal{I} \rightarrow \mathcal{A}$ of the degree map, a global section $1 \in \mathcal{A}(X)$, an addition map $\mathcal{A} \times_{\mathcal{I}} \mathcal{A} \rightarrow \mathcal{A}$ compatible with the maps to \mathcal{I} , and a multiplication map $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \xrightarrow{\cdot} & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{I} \times \mathcal{I} & \xrightarrow{+} & \mathcal{I}, \end{array}$$

satisfying the usual ring identities when either side is defined. (In other words, we only allow addition of sections of equal degree, and require that multiplication adds degrees. Also, for each section $i \in \mathcal{I}$, there is a corresponding zero section in $\mathcal{A}_i := \deg^{-1}(i)$.) Also, if \mathcal{J} is a sheaf of \mathcal{I} -sets, and \mathcal{A} is an \mathcal{I} -indexed ring, we have a similar definition of \mathcal{J} -indexed \mathcal{A} -modules. A \mathcal{J} -indexed homomorphism of \mathcal{J} -indexed \mathcal{A} -modules is then a map $\mathcal{E} \rightarrow \mathcal{F}$ which is \mathcal{A} -linear in the usual sense and which respects degrees. See [Lor00] for more details.

2.1 Basics

Let X be a topological space, \mathcal{I} a sheaf of abelian groups on X , and \mathcal{A} an \mathcal{I} -indexed ring on X . In order to develop the theory of indexed Azumaya algebras, we will need to allow homomorphisms which shift degree.

DEFINITION 2.1. Let \mathcal{J} be a sheaf of \mathcal{I} -sets.

- i) Let \mathcal{E} be a \mathcal{J} -indexed \mathcal{A} -module, and $j \in \mathcal{J}$ a section. Then $\mathcal{E}(j) := \mathcal{E} \times_{\mathcal{J}} \mathcal{I}$, where the map $\mathcal{I} \rightarrow \mathcal{J}$ is addition of j . This is an \mathcal{I} -indexed object via the projection to \mathcal{I} , and an \mathcal{A} -module via the action on \mathcal{E} . (For $i \in \mathcal{I}$, we have $\mathcal{E}(j)_i \simeq \mathcal{E}_{i+j}$. Thus, if $\mathcal{J} = \mathcal{I}$ with the standard action, this agrees with the usual definition.)
- ii) Let \mathcal{E} and \mathcal{F} be \mathcal{I} and \mathcal{J} -indexed \mathcal{A} -modules, respectively. Then $\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is the \mathcal{J} -indexed \mathcal{A} -module such that a section on an open set $U \subseteq X$ consists of a section $j \in \mathcal{J}(U)$ and an \mathcal{I} -indexed \mathcal{A} -linear homomorphism $\phi : \mathcal{E}|_U \rightarrow (\mathcal{F}|_U)(j)$, with the obvious restriction maps, and the degree map of projection to j .

Now if \mathcal{A} is commutative and M is a locally free \mathcal{I} -indexed \mathcal{A} -module of finite rank (with generators not necessarily of degree zero), then $\mathcal{E} := \mathcal{E}nd_{\mathcal{A}}(M)$ is an \mathcal{A} -algebra with center \mathcal{A} . In fact, we have the following result:

THEOREM 2.2. Let \mathcal{J} be a sheaf of \mathcal{I} -sets, and let M and \mathcal{E} be as above.

- i) The functor $E \mapsto M \otimes E$ induces an equivalence of categories from the category of \mathcal{J} -indexed \mathcal{A} -modules to the category of \mathcal{J} -indexed left \mathcal{E} -modules, with quasi-inverse $F \mapsto \mathcal{H}om_{\mathcal{E}}(M, F)$.
- ii) If M' is another locally free \mathcal{A} -module of the same rank as M , with a structure of left \mathcal{E} -module, then the natural map $\mathcal{E} \rightarrow \mathcal{E}nd_{\mathcal{A}}(M')$ is an isomorphism.

Proof. We have natural \mathcal{J} -indexed maps

$$\begin{aligned} \eta_E : E &\rightarrow \mathcal{H}om_{\mathcal{E}}(M, M \otimes E), e \mapsto (m \mapsto m \otimes e) \\ \epsilon_F : M \otimes \mathcal{H}om_{\mathcal{E}}(M, F) &\rightarrow F, m \otimes \phi \mapsto \phi(m). \end{aligned}$$

It is sufficient to show these maps are isomorphisms locally; thus, let $\{m_1, \dots, m_r\}$ be a basis for M , and let $d_i = \deg(m_i) \in \mathcal{I}$. Also, let $\alpha_{ij} \in \mathcal{E}$ be the homomorphism which sends m_k to $\delta_{jk} m_i$. Note that $\deg(\alpha_{ij}) = d_i - d_j$.

Now η_E is clearly injective. For surjectivity, let $\phi \in \mathcal{H}om_{\mathcal{E}}(M, M \otimes E)$. Suppose $\phi(m_1) = \sum_{i=1}^r m_i \otimes e_i$; then since $\alpha_{i1}m_1 = m_i$, we must have $\phi(m_i) = \alpha_{i1}\phi(m_1) = m_i \otimes e_1$ for each i . Therefore, $\phi(m) = m \otimes e_1$ for each $m \in M$, so $\phi = \eta_E(e_1)$.

To show ϵ_F is an isomorphism, we define ϵ_F^{-1} as follows: for each i , let \mathcal{E}_i denote the sub \mathcal{A} -module of \mathcal{E} generated by $\alpha_{1i}, \dots, \alpha_{ri}$. Then we have an isomorphism of left \mathcal{E} -modules $M \rightarrow \mathcal{E}_i(-d_i)$ which sends m_j to α_{ji} . For $f \in F$ of degree $d \in \mathcal{J}$, we now let $\phi_i(f) \in \mathcal{H}om_{\mathcal{E}}(M, F)$ be the composition

$$M \xrightarrow{\sim} \mathcal{E}_i(-d_i) \xrightarrow{f} F(-d_i + d),$$

and define $\epsilon_F^{-1}(f) = \sum_{i=1}^r m_i \otimes \phi_i(f)$. Again, this is a \mathcal{J} -indexed map. To see it is indeed the inverse, note that any element of $M \otimes \mathcal{H}om_{\mathcal{E}}(M, F)$ can be written uniquely as a sum $\sum_{i=1}^r m_i \otimes \psi_i$. This gets mapped by ϵ_F to $f := \sum_i \psi_i(m_i) \in F$; applying the above construction, we see that $\phi_i(f)$ sends m_j to $\alpha_{ji} \sum_k \psi_k(m_k) = \sum_k \psi_k(\alpha_{ji}m_k) = \psi_i(m_j)$, so $\phi_i(f) = \psi_i$. For the other direction, we see that ϵ_F maps $\sum_{i=1}^r m_i \otimes \phi_i(f)$ to $\sum_i \phi_i(f)(m_i) = \sum_i \alpha_{ii} \cdot f = 1 \cdot f$. This completes the proof of (i).

For (ii), since M and M' are both locally free of rank r , it suffices to show that the localized map $\text{End}_{\mathcal{A}_x}(M_x) \rightarrow \text{End}_{\mathcal{A}_x}(M'_x)$ is an isomorphism for each $x \in X$; thus, we may assume X is a one-point space, and M and M' are free. Also, by localizing at an arbitrary \mathcal{I} -indexed prime ideal of \mathcal{A} , we may assume that \mathcal{A} is local. Now by (1) we have an isomorphism $M \otimes N \rightarrow M'$, where $N = \text{Hom}_{\mathcal{E}}(M, M')$. Since M is free, this implies N is a projective \mathcal{A} -module. Also, applying α_{11} to both sides of the isomorphism $M \otimes N \rightarrow M'$ gives an isomorphism $N(-d_1) \simeq (\mathcal{A} \cdot m_1) \otimes N \rightarrow \alpha_{11}M'$, so N is finitely generated. Now the standard proof that finitely generated projective modules over a local ring are free extends to the indexed case, so we see that N is free. Since M and M' are both of rank r , N must have rank 1. Therefore, if N has a free generator of degree d , then $M' \simeq M(-d)$, and the result follows. \square

(In fact, this proof shows that if \mathcal{A}_x is local as an \mathcal{I}_x -indexed ring for each $x \in X$, then locally $M' \simeq M(i)$ as left \mathcal{E} -modules for some $i \in \mathcal{I}$.)

Remark 2.3. It is easy to check that assuming only that M is an \mathcal{E} -module, where \mathcal{E} is an \mathcal{A} -algebra, the natural transformations η and ϵ described in the above proof form the unit and counit, respectively, of an adjunction which makes $M \otimes \cdot$ the left adjoint of $\mathcal{H}om_{\mathcal{E}}(M, \cdot)$.

DEFINITION 2.4. Let \mathcal{A} be a commutative \mathcal{I} -indexed ring, and \mathcal{E} an \mathcal{I} -indexed \mathcal{A} -algebra. For a commutative \mathcal{I} -indexed \mathcal{A} -algebra \mathcal{B} , we say \mathcal{E} *splits over* \mathcal{B} with splitting module M if $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{B} \simeq \mathcal{E}nd_{\mathcal{B}}(M)$ for some locally free \mathcal{I} -indexed \mathcal{B} -module M . We say \mathcal{E} is an *Azumaya algebra* over \mathcal{A} if there is some commutative \mathcal{I} -indexed \mathcal{A} -algebra \mathcal{B} , faithfully flat over \mathcal{A} , such that \mathcal{E} splits over \mathcal{B} .

COROLLARY 2.5. *If \mathcal{E} is an Azumaya algebra over \mathcal{A} of rank r^2 , and there exists a locally free \mathcal{I} -indexed \mathcal{A} -module M of rank r with a structure of left \mathcal{E} -module, then \mathcal{E} is split over \mathcal{A} with splitting module M .*

Proof. Let \mathcal{B} be an \mathcal{A} -algebra, faithfully flat over \mathcal{A} , over which \mathcal{E} splits, and let M' be a splitting module. Then $M \otimes_{\mathcal{A}} \mathcal{B}$ has a structure of left $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{B}$ -module, and it is a locally free \mathcal{B} -module of rank r . On the other hand, if M' has rank s , then $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{B} \simeq \mathcal{E}nd_{\mathcal{B}}(M')$ must have rank s^2 over \mathcal{B} , so $s = r$. Thus, by (ii) of theorem 2.2, the natural map $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{B} \rightarrow \mathcal{E}nd_{\mathcal{B}}(M \otimes_{\mathcal{A}} \mathcal{B}) \simeq \mathcal{E}nd_{\mathcal{A}}(M) \otimes_{\mathcal{A}} \mathcal{B}$ is an isomorphism. Since \mathcal{B} is faithfully flat over \mathcal{A} , this implies that the natural map $\mathcal{E} \rightarrow \mathcal{E}nd_{\mathcal{A}}(M)$ is an isomorphism. \square

Note that since the transpose gives an isomorphism $\mathcal{E}^{\text{op}} \rightarrow \mathcal{E}nd_{\mathcal{A}}(\check{M})$, where $\check{M} = \mathcal{H}om_{\mathcal{A}}(M, \mathcal{A})$, the above theory works equally well for right \mathcal{E} -modules.

2.2 Indexed Algebras Associated to a Log Structure

Recall from [Lor00] that to any log scheme X we may associate a canonical $\overline{\mathcal{M}}_X^{\text{gp}}$ -indexed \mathcal{O}_X -algebra $\mathcal{A}_X^{\text{gp}}$, which corresponds to the exact sequence of abelian groups $0 \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{M}_X^{\text{gp}} \rightarrow \overline{\mathcal{M}}_X^{\text{gp}} \rightarrow 0$. In particular, for $\bar{s} \in \overline{\mathcal{M}}_X^{\text{gp}}$, $(\mathcal{A}_X^{\text{gp}})_{\bar{s}}$ is the invertible sheaf on X corresponding to the \mathcal{O}_X^* -torsor given by the inverse image of \bar{s} in $\mathcal{M}_X^{\text{gp}}$; thus, for a section $s \in \mathcal{M}_X^{\text{gp}}$, we have a corresponding basis element e_s of $(\mathcal{A}_X^{\text{gp}})_{\bar{s}}$, where \bar{s} is the image of s in $\overline{\mathcal{M}}_X^{\text{gp}}$. We then have a canonical connection d on $\mathcal{A}_X^{\text{gp}}$ characterized by the formula $de_s = e_s \otimes d \log s$, and we define $\mathcal{B}_{X/S}$ to be the kernel of d . Note that since d is multiplicative, $\mathcal{B}_{X/S}$ is a subring of $\mathcal{A}_X^{\text{gp}}$.

The construction of $\mathcal{A}_X^{\text{gp}}$ is functorial in the following way: given a morphism $f : X \rightarrow Y$ of fine log schemes, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & (f^* \mathcal{M}_Y)^{\text{gp}} & \longrightarrow & f^{-1} \overline{\mathcal{M}}_Y^{\text{gp}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & \mathcal{M}_X^{\text{gp}} & \longrightarrow & \overline{\mathcal{M}}_X^{\text{gp}} \longrightarrow 0. \end{array}$$

This induces a natural cartesian diagram

$$\begin{array}{ccc} f^* \mathcal{A}_Y^{\text{gp}} & \longrightarrow & \mathcal{A}_X^{\text{gp}} \\ \downarrow & & \downarrow \\ f^{-1} \overline{\mathcal{M}}_Y^{\text{gp}} & \longrightarrow & \overline{\mathcal{M}}_X^{\text{gp}}. \end{array}$$

For $s \in \mathcal{M}_Y^{\text{gp}}$, the map $f^* \mathcal{A}_Y^{\text{gp}} \rightarrow \mathcal{A}_X^{\text{gp}}$ sends $1 \otimes e_s$ to e_{f^*s} .

Now recall the relative Frobenius diagram for log schemes:

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/S}} & X' & & \\ & \searrow & \downarrow & \searrow \pi_{X/S} & \\ & & X'' & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ & & S & \xrightarrow{F_S} & S. \end{array}$$

Here the bottom square is cartesian in the category of fine log schemes, and $F_{X/S}$ is uniquely determined by the requirement that $F_{X/S}$ is purely inseparable, and the map $X' \rightarrow X''$ is étale. Note that our notation for X' and X'' is opposite that used by Kato in [Kat88] but agrees with [Ogu04]. We then have the Cartier isomorphisms $C_{X/S} : \mathcal{H}^q(F_{X/S*} \Omega_{X/S}) \xrightarrow{\sim} \Omega_{X'/S}^q$ of $\mathcal{O}_{X'}$ -modules. In fact, we have the following generalization:

PROPOSITION 2.6. *Assume $X \rightarrow S$ is a smooth morphism of fine schemes. Then there are canonical isomorphisms*

$$\mathcal{H}^i(\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \Omega_{X/S}, d) \simeq \mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}^i,$$

where $(\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \Omega_{X/S}, d)$ is the de Rham complex of $\mathcal{A}_X^{\text{gp}}$ with the canonical connection d .

Proof. (This is a straightforward generalization of the proof in [Kat88, 4.12].) We have a natural map $\mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}^i \rightarrow \mathcal{H}^i(\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \Omega_{X/S}, d)$ which extends the inverse Cartier isomorphism $C_{X/S}^{-1} : \Omega_{X'/S}^i \rightarrow \mathcal{H}^i(\Omega_{X/S}, d)$. Showing this map is an isomorphism is an étale local problem, so we may assume we have a chart $(Q \rightarrow \mathcal{M}_S, P \rightarrow \mathcal{M}_X, Q \rightarrow P)$ of $X \rightarrow S$ such that P and Q are finitely

generated integral monoids, the map $Q^{\text{gp}} \rightarrow P^{\text{gp}}$ is injective and the torsion part of its cokernel is a finite group of order prime to p , and the natural map $X \rightarrow S \times_{\mathbb{F}_p[Q]} \mathbb{F}_p[P]$ is an isomorphism. Then $X' \simeq S \times_{\mathbb{F}_p[Q]} \mathbb{F}_p[H]$, where $H := P \cap (pP^{\text{gp}} + Q^{\text{gp}})$.

Now for $s \in P^{\text{gp}}$, let \bar{s} be the image of s in $\overline{\mathcal{M}}_X^{\text{gp}}$; then using e_s as a basis element for $(\mathcal{A}_X^{\text{gp}})_{\bar{s}}$, the degree \bar{s} part of $(\mathcal{A}_X^{\text{gp}} \otimes \Omega_{X/S}, d)$ is isomorphic to the de Rham complex $(\Omega_{X/S}, d + d \log s \wedge)$. We can now decompose this complex as a direct sum of complexes C_ν for $\nu \in \mathbb{F}_p \otimes (P^{\text{gp}}/Q^{\text{gp}}) \simeq P^{\text{gp}}/H^{\text{gp}}$, where

$$C_\nu^i := \mathcal{O}_S \otimes_{\mathbb{F}_p[Q]} \mathbb{F}_p[P \cap \nu] \otimes_{\mathbb{F}_p} \wedge^i(\mathbb{F}_p \otimes (P^{\text{gp}}/Q^{\text{gp}}))$$

with the differential $a \mapsto (s + \nu) \wedge a$. (Here $\mathbb{F}_p[P \cap \nu]$ denotes the submodule of $\mathbb{F}_p[P]$ generated by elements of $P \cap \nu$.) This complex is exact if $s + \nu \neq 0$, while if $s + \nu = 0$, then $\mathcal{H}^i(C_\nu) \simeq C_\nu^i \simeq C_\nu^0 \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}^i$. Hence $\mathcal{H}^i(\Omega_{X/S}, d + d \log s \wedge) \simeq \mathcal{H}^0(\Omega_{X/S}, d + d \log s \wedge) \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}^i$. However, $\mathcal{H}^0(\Omega_{X/S}, d + d \log s \wedge)$ is the kernel of $d + d \log s$, which by definition corresponds to $(\mathcal{B}_{X/S})_{\bar{s}}$. Since the image of P^{gp} generates $\overline{\mathcal{M}}_X^{\text{gp}}$, we are done. \square

Remark 2.7. Since $(\mathcal{A}_X^{\text{gp}} \otimes \Omega_{X/S}, d)$ has zero p -curvature, the Higgs complex is $(\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}, 0)$, and the kernel of the natural connection on $\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}^i$ is $\mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}^i$. Thus, this is a special case of [Ogu04, 3.1.1], which states that $\mathcal{H}^i(E \otimes \Omega_{X/S}, \nabla) \simeq \mathcal{H}^i(E \otimes F_{X/S}^* \Omega_{X'/S}, -\psi)^\nabla$. Similarly, for $i < p$, this is a special case of the main result of chapter 4, since in notation yet to be developed we will have $C_{X/S}(\mathcal{A}_X^{\text{gp}}, d) \simeq (\mathcal{B}_{X/S}, 0)$.

The proof of this isomorphism also gives us insight into the structure of $\mathcal{B}_{X/S}$. Namely, we see that given a local chart $(Q \rightarrow \mathcal{M}_S, P \rightarrow \mathcal{M}_X, Q \rightarrow P)$ as in the proof, then for $m \in P^{\text{gp}}$,

$$(\mathcal{B}_{X/S})_m = e_m \mathcal{O}_S \otimes_{\mathbb{F}_p[Q]} \mathbb{F}_p[(-m + H^{\text{gp}}) \cap P] \subseteq e_m \mathcal{O}_S \otimes_{\mathbb{F}_p[Q]} \mathbb{F}_p[P] = (\mathcal{A}_X^{\text{gp}})_m.$$

This immediately gives the following:

COROLLARY 2.8. *The natural map $\mathcal{A}_{X'}^{\text{gp}} \rightarrow F_{X/S*} \mathcal{A}_X^{\text{gp}}$ factors through $F_{X/S*} \mathcal{B}_{X/S}$. Furthermore, $F_{X/S*} \mathcal{B}_{X/S}$ is generated as an $F_{X/S*} \overline{\mathcal{M}}_X^{\text{gp}}$ -indexed module over $\mathcal{A}_{X'}^{\text{gp}}$ by elements of the form $\alpha(m)e_{-m}$ for $m \in P$.*

Proof. Locally, $\overline{\mathcal{M}}_{X'}^{\text{gp}}$ is generated by H^{gp} , and for $m \in H^{\text{gp}}$, we have $e_m \mathcal{O}_{X'} = e_m \mathcal{O}_S \otimes_{\mathbb{F}_p[Q]} \mathbb{F}_p[H^{\text{gp}}] \subseteq \mathcal{B}_{X/S}$. On the other hand, the above description of $\mathcal{B}_{X/S}$ shows that it is generated as an indexed \mathcal{O}_S -module by elements of the form $\alpha(-m+h)e_m$, where $h \in H^{\text{gp}}$ is such that $-m+h \in P$. However, then $\alpha(-m+h)e_m = e_h[\alpha(-m+h)e_{-(m+h)}]$, and $e_h \in \mathcal{A}_{X'}^{\text{gp}}$. \square

Example 2.9. Consider the monoid $P = \langle a, b, c : a + b = 2c \rangle$, and let $X := \text{Spec}(P \rightarrow k[P])$ for some field k of characteristic p . We then claim that $F_{X/k}$ is not flat. To see this, we can identify P with $\{(m, n) \in \mathbb{Z}^2 : m \geq |n|\}$ by sending a to $(1, 1)$, b to $(1, -1)$, and c to $(1, 0)$. We then see that $\{e^m : m \in P - PH^+\}$ forms a basis for $k[P]/k[PH^+]$ as a vector space over $k[H]/k[H^+] \simeq k$. However, for some cosets S of $H^{\text{gp}} = p\mathbb{Z}^2$, $S \cap P$ has more than one minimal element; for example, $((-1, -1) + H^{\text{gp}}) \cap P$ has minimal elements $(p-1, p-1)$ and $(p-1, -1)$. Therefore, $k[P] \otimes_{k[H]} k$ has dimension strictly greater than p^2 . But localizing to $D(H^+)$, we get $k[P^{\text{gp}}]$ over $k[H^{\text{gp}}]$, which is free of rank p^2 , since choosing a set of representatives S of $P^{\text{gp}}/H^{\text{gp}}$, we get a basis $\{e^s : s \in S\}$ of $k[P^{\text{gp}}]$.

On the other hand, the sets $S \cap P$ for S a coset of H^{gp} form a partition of P into p^2 sets. Therefore, if we choose a set of representatives S of $P^{\text{gp}}/H^{\text{gp}}$, then $\{e_s : s \in S\}$ forms a basis for $\mathcal{A}_X^{\text{gp}}$ as a $\mathcal{B}_{X/S}$ -module (since $-S$ is also a set of representatives).

Generalizing this argument, we get:

PROPOSITION 2.10. (See also [Lor00, 2.6].) Let $f : X \rightarrow S$ be a smooth morphism of log schemes of characteristic p , with X of relative dimension r over S . Then $\mathcal{A}_X^{\text{gp}}$ is locally free of rank p^r as a $\mathcal{B}_{X/S}$ -module. In fact, if $m_1, \dots, m_r \in \mathcal{M}_X^{\text{gp}}$ is a logarithmic system of coordinates, then letting $\theta_i := e_{m_i}$ and $\theta^I := \prod_{k=1}^r \theta_k^{I_k}$ for $I \in \mathbb{Z}^r$, then $\{\theta^I : I \in \{0, \dots, p-1\}^r\}$ forms a basis.

Proof. Locally, if we have a chart $(Q \rightarrow \mathcal{M}_S, P \rightarrow \mathcal{M}_X, Q \rightarrow P)$, we get as in the example above that $\mathcal{A}_X^{\text{gp}}$ is free over $\mathcal{B}_{X/S}$. All we need to show is that $P^{\text{gp}}/H^{\text{gp}}$ has p^r elements. However, $P^{\text{gp}}/H^{\text{gp}} \simeq \mathbb{F}_p \otimes (P^{\text{gp}}/Q^{\text{gp}})$ is a vector space over \mathbb{F}_p , and when we base extend to \mathcal{O}_X we get $\Omega_{X/S}^1$, which is locally free of rank r . Therefore, $\mathbb{F}_p \otimes (P^{\text{gp}}/Q^{\text{gp}})$ has dimension r , and the result follows. Given a logarithmic system of coordinates, we may choose the chart so that m_1, \dots, m_r are in the image of $P^{\text{gp}} \rightarrow \mathcal{M}_X^{\text{gp}}$, in which case $\{\sum_{k=1}^r I_k m_k : I \in \{0, \dots, p-1\}^r\}$ forms a set of representatives of $P^{\text{gp}}/H^{\text{gp}}$. \square

2.3 The Azumaya Algebra $\tilde{\mathcal{D}}_{X/S}$

Our main application of this theory will be to an extension of the ring of PD differential operators on X , where $f : X \rightarrow S$ is a smooth morphism of log schemes of characteristic p . We briefly review the standard notation: first, $D(1)$ is the logarithmic PD envelope of the diagonal embedding $\Delta : X \rightarrow X \times_S X$, and $\mathcal{D}(1) := \mathcal{O}_{D(1)}$, which is considered to be an \mathcal{O}_X -algebra via the projection $p_1 : D(1) \rightarrow X$. Then for $m \in \mathcal{M}_X^{\text{gp}}$, $\eta_m \in \mathcal{D}(1)$ is the unique element of the PD ideal in $\mathcal{D}(1)$ such that $1 + \eta_m = \alpha_{D(1)}(p_2^* m - p_1^* m)$. If $m_1, \dots, m_r \in \mathcal{M}_X^{\text{gp}}$ is a logarithmic system of coordinates (i.e. $d \log m_1, \dots, d \log m_r$ form a basis for $\Omega_{X/S}^1$), then $\{D_I : I \in \mathbb{N}^r\}$ denotes the basis of the ring of PD differential operators dual to the basis $\{\eta^{[I]} = \prod_{k=1}^r \eta_{m_k}^{[I_k]} : I \in \mathbb{N}^r\}$ of $\mathcal{D}(1)$. (In this paper we will not be using the alternate basis $\{\zeta^{[I]}\}$ of $\mathcal{D}(1)$, where $\zeta_m := \log(1 + \eta_m)$, except in the appendix; this basis has the property that the dual basis $\{D^I : I \in \mathbb{N}^r\}$ satisfies $D^I \circ D^J = D^{I+J}$.)

DEFINITION 2.11. Let $\mathcal{D}_{X/S}$ denote the ring of PD differential operators on X relative to S . Then $\tilde{\mathcal{D}}_{X/S}$ is the $\overline{\mathcal{M}}_X^{\text{gp}}$ -indexed \mathcal{O}_X -algebra $\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}$, given the unique multiplication such that $\mathcal{A}_X^{\text{gp}} \rightarrow \tilde{\mathcal{D}}_{X/S}$, $a \mapsto a \otimes 1$ and $\mathcal{D}_{X/S} \rightarrow \tilde{\mathcal{D}}_{X/S}$, $\phi \mapsto 1 \otimes \phi$ are ring homomorphisms, $(a \otimes 1)(1 \otimes \phi) = a \otimes \phi$, and for each log derivation ∂ ,

$$(1 \otimes \partial)(a \otimes 1) = (\partial.a) \otimes 1 + a \otimes \partial.$$

(Here $\partial.a$ denotes the evaluation of ∂ at a as opposed to composition of PD differential operators, where we give $\mathcal{A}_X^{\text{gp}}$ the $\mathcal{D}_{X/S}$ -module structure corresponding to the canonical connection d on $\mathcal{A}_X^{\text{gp}}$.)

A straightforward induction shows that in local coordinates, $(1 \otimes D_N)(a \otimes 1) = \sum_{I \leq N} \binom{N}{I} (D_{N-I}.a) \otimes D_I$. Thus, this agrees with the definition of $\tilde{\mathcal{D}}_{X,\mathcal{I}}^{(0)}$ from [Mon02].

For notational convenience, we will often treat the given maps $\mathcal{A}_X^{\text{gp}} \rightarrow \tilde{\mathcal{D}}_{X/S}$, $\mathcal{D}_{X/S} \rightarrow \tilde{\mathcal{D}}_{X/S}$ as embeddings, and thus write $a\phi$ in place of $a \otimes \phi$. We now have an inclusion

$$\tilde{\mathcal{D}}_{X/S} \hookrightarrow \text{PD Diff}(\mathcal{A}_X^{\text{gp}}, \mathcal{A}_X^{\text{gp}}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}(1) \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}, \mathcal{A}_X^{\text{gp}}),$$

which sends a section $a \in \mathcal{A}_X^{\text{gp}}$ to multiplication by a and is compatible with the map $\mathcal{D}_{X/S} \rightarrow \text{PD Diff}(\mathcal{A}_X^{\text{gp}}, \mathcal{A}_X^{\text{gp}})$ corresponding to the canonical connection d . (The relation $\partial \circ a = (\partial.a) + a\partial$ holds in $\text{PD Diff}(\mathcal{A}_X^{\text{gp}}, \mathcal{A}_X^{\text{gp}})$ since d is multiplicative.) In fact, it is not hard to see that the image is exactly the set of PD differential operators ϕ such that $\phi(\tau \otimes e_s) = \phi(\tau(1 + \eta_s)) \cdot e_s$ for $\tau \in \mathcal{D}(1)$, $s \in \mathcal{M}_X^{\text{gp}}$.

The following result indicates the significance of the ring $\tilde{\mathcal{D}}_{X/S}$. Recall from [Lor00] that a connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^1$ on a \mathcal{J} -indexed $\mathcal{A}_X^{\text{gp}}$ -module E is *admissible* if and only if $\nabla(ae) = a\nabla(e) + e \otimes da$ whenever $a \in \mathcal{A}_X^{\text{gp}}$, $e \in \mathcal{E}$.

PROPOSITION 2.12. *Let \mathcal{J} be a sheaf of $\overline{\mathcal{M}}_X^{\text{gp}}$ -sets, and let \mathcal{E} be a \mathcal{J} -indexed $\mathcal{A}_X^{\text{gp}}$ -module with integrable connection ∇ on \mathcal{E} . Then ∇ is admissible if and only if the $\mathcal{A}_X^{\text{gp}}$ -module structure on \mathcal{E} and the $\mathcal{D}_{X/S}$ -module structure on \mathcal{E} corresponding to ∇ extend to a $\tilde{\mathcal{D}}_{X/S}$ -module structure on \mathcal{E} .*

Proof. First, note that if we do have such an extension, it must be given by $(a \otimes \phi)e = a\nabla_\phi e$. Now admissibility of ∇ is equivalent to the condition that for every log derivation ∂ on X , $\nabla_\partial(ae) = a\nabla_\partial(e) + \partial(a)e$. Rewriting, this tells us that

$$(1 \otimes \partial)[(a \otimes 1)e] = (\partial \cdot a \otimes 1 + a \otimes \partial)e = [(1 \otimes \partial)(a \otimes 1)]e.$$

From this the desired equivalence easily follows. \square

We can also provide a simple crystalline interpretation of the category of $\tilde{\mathcal{D}}_{X/S}$ -modules. In particular, if $g : T_1 \rightarrow T_2$ is a morphism in $\text{Crys}(X/S)$, then the natural map $g^{-1}\overline{\mathcal{M}}_{T_2}^{\text{gp}} \rightarrow \overline{\mathcal{M}}_{T_1}^{\text{gp}}$ is an isomorphism, so $g^*\mathcal{A}_{T_2}^{\text{gp}} \rightarrow \mathcal{A}_{T_1}^{\text{gp}}$ is also an isomorphism. Hence the functor $T \mapsto \mathcal{A}_T^{\text{gp}}$ gives a crystal of $\overline{\mathcal{M}}_X^{\text{gp}}$ -indexed $\mathcal{O}_{X/S}$ -algebras. Now letting T be the logarithmic PD envelope of the diagonal in $X \times_S X$ with the two projections $p_1, p_2 : T \rightarrow X$, for $s \in \mathcal{M}_X^{\text{gp}}$ we calculate

$$p_2^*(e_s) = e_{(0,s)} = e_{(s,0)}e_{(-s,s)} = p_1^*(e_s)(1 + \eta_s).$$

(Here $(-s, s) \in \mathcal{M}_T^*$, so $e_{(-s,s)} = \alpha_T(-s, s) = 1 + \eta_s$.) Thus the HPD stratification on $\mathcal{A}_X^{\text{gp}}$ sends $1 \otimes e_s$ to $e_s \otimes (1 + \eta_s)$, and in particular, the connection this crystal defines on $\mathcal{A}_X^{\text{gp}}$ agrees with d . It is then easy to show:

PROPOSITION 2.13. *Suppose we have a locally nilpotent \mathcal{J} -indexed $\mathcal{A}_X^{\text{gp}}$ -module E with connection ∇ . Then ∇ is admissible if and only if in the corresponding crystal of \mathcal{J} -indexed $\mathcal{O}_{X/S}$ -modules, the transition maps $\theta_g : g^*E_{T_2} \rightarrow E_{T_1}$ are $g^*\mathcal{A}_{T_2}^{\text{gp}}$ -linear.*

Hence the category of locally nilpotent \mathcal{J} -indexed $\tilde{\mathcal{D}}_{X/S}$ -modules is equivalent to the category of crystals of \mathcal{J} -indexed $\mathcal{A}_X^{\text{gp}}$ -modules on $\text{Crys}(X/S)$.

THEOREM 2.14. *Let $\mathfrak{Z} = S(\mathcal{T}_{X'/S})$, which we embed into $\mathcal{D}_{X/S}$ via the map $\partial \mapsto \partial^p - \partial^{(p)}$. Let $\tilde{\mathfrak{Z}}$ be the center of $\tilde{\mathcal{D}}_{X/S}$. Then $\tilde{\mathfrak{Z}} = \mathcal{B}_{X/S} \otimes_{\mathcal{O}_X} \mathfrak{Z}$ as a subring of $\tilde{\mathcal{D}}_{X/S}$.*

Proof. If $f \in \mathcal{B}_{X/S}$, then since $\partial f = 0$ for any log derivation ∂ , we see that f commutes with $\mathcal{D}_{X/S}$, and it obviously commutes with $\mathcal{A}_X^{\text{gp}}$; thus, since $\mathcal{D}_{X/S}$ and $\mathcal{A}_X^{\text{gp}}$ generate $\tilde{\mathcal{D}}_{X/S}$ as a ring, we get $f \in \tilde{\mathfrak{Z}}$.

Now choose a system of logarithmic coordinates $m_1, \dots, m_r \in \mathcal{M}_X^{\text{gp}}$, and let $\theta_i = e_{m_i}$. Finally, let $M = \{0, 1, \dots, p-1\}^r$. Then $\mathcal{A}_X^{\text{gp}}$ is a locally free $\mathcal{B}_{X/S}$ -module with local basis $\{\theta^I : I \in M\}$. Thus, $\tilde{\mathcal{D}}_{X/S}$ is generated as a $\mathcal{B}_{X/S}$ -algebra by $\{\theta_i, D_{\epsilon_i}\}$. Expressing a section $\phi \in \tilde{\mathcal{D}}_{X/S}$ as a sum $\sum_N f_N D_N$, and recalling that $\phi^b(1, e_s) = \phi^b(s, 1)e_s$ for $\phi \in \mathcal{D}_{X/S}$ and $s \in \mathcal{M}_X^{\text{gp}}$ [Mon02, 4.1.1], we now calculate

$$\begin{aligned} [\phi, \theta_i] &= \sum_N N_i f_N \theta_i D_{N-\epsilon_i}; \\ [\phi, D_{\epsilon_i}] &= - \sum_N (D_{\epsilon_i} f_N) \cdot D_N. \end{aligned}$$

Therefore, $\phi \in \tilde{\mathfrak{Z}}$ if and only if $f_N \neq 0$ only for $p \mid N$, and $df_N = 0$ for each N , i.e. $f_N \in \mathcal{B}_{X/S}$. Since the embedding $\mathfrak{Z} \hookrightarrow \mathcal{D}_{X/S}$ sends D_{ϵ_i} to $D_{\epsilon_i}^p - D_{\epsilon_i} = D_{p\epsilon_i}$, so that its image is generated as an $\mathcal{O}_{X'}$ -module by $\{D_{pN} : N \in \mathbb{N}^r\}$, this completes the proof. \square

Note that we also get that the centralizer of $\mathcal{A}_X^{\text{gp}}$ is $\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \mathfrak{Z}$. Denote this by \mathcal{C}_X , and consider $\tilde{\mathcal{D}}_{X/S}$ as a right \mathcal{C}_X -module by multiplication on the right.

THEOREM 2.15. *The map $\tilde{\mathcal{D}}_{X/S} \otimes_{\tilde{\mathfrak{Z}}} \mathcal{C}_X \rightarrow \mathcal{E}nd_{\mathcal{C}_X}(\tilde{\mathcal{D}}_{X/S})$ by multiplication on the left by $\tilde{\mathcal{D}}_{X/S}$ and on the right by \mathcal{C}_X is an isomorphism.*

Proof. It suffices to work locally, so choose a local system of logarithmic coordinates m_1, \dots, m_r . Then locally, $\tilde{\mathcal{D}}_{X/S} \otimes_{\tilde{\mathfrak{Z}}} \mathcal{C}_X$ has a basis $\{1 \otimes \theta^I : I \in M = \{0, 1, \dots, p-1\}^r\}$ as a left $\tilde{\mathcal{D}}_{X/S}$ -module. Also, $\mathcal{E}nd_{\mathcal{C}_X}(\tilde{\mathcal{D}}_{X/S})$ has a basis $\{\alpha_I : I \in M\}$ as a left $\tilde{\mathcal{D}}_{X/S}$ -module, where α_I is the unique homomorphism which sends D_J to δ_{IJ} . (This is because $\tilde{\mathcal{D}}_{X/S}$ has a basis $\{D_I : I \in M\}$ as a right \mathcal{C}_X -module.)

We now calculate that $1 \otimes \theta_i$ acts on $\tilde{\mathcal{D}}_{X/S}$ by sending D_N to $\theta_i D_N + N_i \theta_i D_{N-\epsilon_i}$. Thus, setting $\beta_i = \theta_i^{-1} \otimes \theta_i - 1 \otimes 1$, β_i acts by sending D_N to $N_i D_{N-\epsilon_i}$. Now letting $\beta^K := \prod_{i=1}^r \beta_i^{K_i}$ for $K \in M$, this implies that β^K acts on $\tilde{\mathcal{D}}_{X/S}$ by sending D_I to $\frac{I!}{(I-K)!} D_{I-K}$ if $I \geq K$, and 0 otherwise.

We thus have

$$\beta^J = \sum_{I \leq J} (-1)^{|J-I|} \binom{J}{I} (\theta^{-I} \otimes \theta^I).$$

Thus, enumerating M in some order compatible with the product partial order, the transition matrix from the set $\{\beta^J\}$ to the basis $\{1 \otimes \theta^I : I \in M\}$ of $\tilde{\mathcal{D}}_{X/S} \otimes_{\tilde{\mathfrak{Z}}} \mathcal{C}_X$ is upper triangular, with units on the diagonal, so $\{\beta^J : J \in M\}$ is also a basis for $\tilde{\mathcal{D}}_{X/S} \otimes_{\tilde{\mathfrak{Z}}} \mathcal{C}_X$. Similarly, letting β^J also denote the corresponding endomorphism on $\tilde{\mathcal{D}}_{X/S}$, we have

$$\beta^J = \sum_{I \geq J, I \in M} \left(\frac{I!}{(I-J)!} D_{I-J} \right) \alpha_I.$$

Thus, the transition matrix from the set $\{\beta^J : J \in M\}$ to the basis $\{\alpha_I : I \in M\}$ of $\mathcal{E}nd_{\mathcal{C}_X}(\tilde{\mathcal{D}}_{X/S})$ is lower triangular with units on the diagonal, so $\{\beta^J : J \in M\}$ is also a basis for $\mathcal{E}nd_{\mathcal{C}_X}(\tilde{\mathcal{D}}_{X/S})$. \square

COROLLARY 2.16. *The indexed ring $\tilde{\mathcal{D}}_{X/S}$ is an Azumaya algebra over its center $\tilde{\mathfrak{Z}}$ of rank p^{2r} , where r is the relative dimension of X over S .*

Proof. Since $\mathcal{A}_X^{\text{gp}}$ is locally free and thus faithfully flat as a $\mathcal{B}_{X/S}$ -module, $\mathcal{C}_X \simeq \mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{B}_{X/S}} \tilde{\mathfrak{Z}}$ is a faithfully flat extension of $\tilde{\mathfrak{Z}}$, and by the previous theorem $\tilde{\mathcal{D}}_{X/S}$ splits over \mathcal{C}_X with splitting module $\tilde{\mathcal{D}}_{X/S}$. Since $\tilde{\mathcal{D}}_{X/S}$ has a basis $\{D_I : I \in \{0, \dots, p-1\}^r\}$ as a right \mathcal{C}_X -module, the rank must be p^r , so $\tilde{\mathcal{D}}_{X/S}$ has rank p^{2r} over $\tilde{\mathfrak{Z}}$. \square

As an application, we can recover the main result of [Lor00] as follows. Consider $\mathcal{B}_{X/S}$ as a $\tilde{\mathfrak{Z}}$ -algebra via base extension of the quotient map $S'(\mathcal{T}_{X'/S}) \rightarrow S'(\mathcal{T}_{X'/S})/S^+(\mathcal{T}_{X'/S}) \simeq \mathcal{O}_{X'}$, and let $\tilde{\mathcal{D}}_0 := \tilde{\mathcal{D}}_{X/S} \otimes_{\tilde{\mathfrak{Z}}} \mathcal{B}_{X/S}$. Then $\mathcal{A}_X^{\text{gp}}$ is a locally free $\mathcal{B}_{X/S}$ -module of rank p^r (compatible with the $\tilde{\mathfrak{Z}}$ -algebra structure of $\mathcal{B}_{X/S}$ since the connection d is p -integrable), which has a structure of left module over $\tilde{\mathcal{D}}_0$. Therefore, $\tilde{\mathcal{D}}_{X/S}$ splits over $\mathcal{B}_{X/S}$, and for \mathcal{I} a sheaf of $\overline{\mathcal{M}}_X^{\text{gp}}$ -sets we get an equivalence between the \mathcal{I} -indexed $\mathcal{B}_{X/S}$ -modules and the \mathcal{I} -indexed left $\tilde{\mathcal{D}}_0$ -modules, or equivalently the \mathcal{I} -indexed $\mathcal{A}_X^{\text{gp}}$ -modules with integrable, p -integrable, admissible connections. This equivalence sends a $\mathcal{B}_{X/S}$ -module \mathcal{F} to $\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{B}_{X/S}} \mathcal{F}$ with the connection acting on $\mathcal{A}_X^{\text{gp}}$, and it sends an $\mathcal{A}_X^{\text{gp}}$ -module \mathcal{E} with connection ∇ to $\mathcal{H}om_{\tilde{\mathcal{D}}_0}(\mathcal{A}_X^{\text{gp}}, \mathcal{E}) \simeq \mathcal{E}^\nabla$.

3. The Fundamental Extension

DEFINITION 3.1. Suppose we are given a morphism $f : X \rightarrow S$ of fine log schemes of characteristic p . Then a *lifting of f modulo p^n* is a map $\tilde{f} : \tilde{X} \rightarrow \tilde{S}$ of fine log schemes flat over \mathbb{Z}/p^n which fits into a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & \tilde{X} \\ f \downarrow & & \downarrow \tilde{f} \\ S & \longrightarrow & \tilde{S}, \end{array}$$

where $S \rightarrow \tilde{S}$ is the closed immersion defined by p .

Note that since $\mathrm{Spec}(\mathbb{Z}/p) \rightarrow \mathrm{Spec}(\mathbb{Z}/p^n)$ is an exact closed immersion, so are the base extensions $X \rightarrow \tilde{X}$ and $S \rightarrow \tilde{S}$. If f is smooth, respectively étale, resp. integral, resp. exact, so is \tilde{f} . We will mostly be interested in liftings modulo p^2 , which is what we will mean if we do not specify n .

For the rest of the paper, we suppose we are given an integral smooth morphism $f : X \rightarrow S$ of fine log schemes of characteristic p , with a given lifting of $X' \rightarrow S$ modulo p^2 , $\tilde{X}' \rightarrow \tilde{S}$. We denote these data by $\mathcal{X}/S := (X \rightarrow S, \tilde{X}' \rightarrow \tilde{S})$. For example, if $f : X \rightarrow S$ has a given lifting $\tilde{f} : \tilde{X} \rightarrow \tilde{S}$, and F_S has a lifting $F_{\tilde{S}} : \tilde{S} \rightarrow \tilde{S}$, we may define \tilde{X}'' to be the pullback of $F_{\tilde{S}}$ and \tilde{f} . Then since $X' \rightarrow X''$ is étale, there is a unique lifting $\tilde{X}' \rightarrow \tilde{X}''$. Alternately, a smooth lifting exists if $H^2(X', \mathcal{T}_{X'/S}) = 0$, in particular if X is affine or if X is a curve over a field k , and since we assume $X \rightarrow S$ integral, a smooth lifting is automatically a lifting in the sense defined above.

As in [OV], we will use a lifting \tilde{X}'/\tilde{S} to construct a canonical sheaf of \mathcal{O}_X -algebras $\mathcal{K}_{\mathcal{X}/S}$ with multiplicative connection, along with a natural filtration N_\bullet on $\mathcal{K}_{\mathcal{X}/S}$ such that $N_i \mathcal{K}_{\mathcal{X}/S} \simeq S^i(N_1 \mathcal{K}_{\mathcal{X}/S})$ and a short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow N_1 \mathcal{K}_{\mathcal{X}/S} \rightarrow F_{X/S}^* \Omega_{X'/S}^1 \rightarrow 0$$

in which the maps are horizontal.

3.1 Torsors over Locally Free Sheaves

In this section we give the general construction which gives rise to $\mathcal{K}_{\mathcal{X}/S}$.

LEMMA 3.2. Let X be a ringed topos, with a locally free \mathcal{O}_X -module \mathcal{T} of finite rank; let $\Omega := \mathcal{T}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}, \mathcal{O}_X)$. Suppose we have a \mathcal{T} -torsor \mathcal{L} .

- i) There exists an \mathcal{O}_X -algebra \mathcal{K} such that \mathcal{L} is isomorphic to the sheaf of \mathcal{O}_X -algebra homomorphisms $\mathcal{K} \rightarrow \mathcal{O}_X$. Furthermore, there is a canonical cotorsor structure on \mathcal{K} consisting of a coaction map $\mathcal{K} \rightarrow \mathcal{K} \otimes S^* \Omega$ and a cosubtraction map $S^* \Omega \rightarrow \mathcal{K} \otimes \mathcal{K}$, such that the action of $D \in \mathcal{T}$ on \mathcal{L} corresponds to composition with the automorphism

$$\mathcal{K} \rightarrow \mathcal{K} \otimes S^* \Omega \xrightarrow{\mathrm{id} \otimes S^* D} \mathcal{K}.$$

- ii) There is a canonical filtration N_\bullet on \mathcal{K} such that letting $\mathcal{E} := N_1 \mathcal{K}$, we have $S^i \mathcal{E} \simeq N_i \mathcal{K}$ via the natural map $S^i \mathcal{E} \rightarrow \mathcal{K}$.
- iii) There is a canonical locally split exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \Omega \rightarrow 0.$$

- iv) If X is the Zariski topos on a scheme, then \mathcal{K} is quasicohherent, and the affine scheme $\mathbf{T}(\mathcal{L}) := \mathbf{Spec} \mathcal{K}$ over X represents the functor \mathcal{L} .
- v) If X is the crystalline topos on a scheme over S , and \mathcal{T} is a crystal of $\mathcal{O}_{X/S}$ -modules, then \mathcal{E} and \mathcal{K} are also crystals of $\mathcal{O}_{X/S}$ -modules and $\mathcal{O}_{X/S}$ -algebras, respectively.

Proof. We first construct \mathcal{E} as the subsheaf of $\mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ consisting of morphisms $\phi : \mathcal{L} \rightarrow \mathcal{O}_X$ such that for any local section a of \mathcal{L} , the function $\mathcal{T} \rightarrow \mathcal{O}_X$, $D \mapsto \phi(a + D) - \phi(a)$, is \mathcal{O}_X -linear. Note that for any object U of X , if this is true for one $a \in \mathcal{L}(U)$, it is true for all $a \in \mathcal{L}(U)$, and in this case, the function is independent of the choice of a .

To get the exact sequence, note that any constant function on \mathcal{L} is in \mathcal{E} ; this gives an injection $\mathcal{O}_X \rightarrow \mathcal{E}$. Also, given $\phi \in \mathcal{E}$, as we remarked above, the map $\mathcal{T} \rightarrow \mathcal{O}_X$, $D \mapsto \phi(a + D) - \phi(a)$, is independent of the choice of a ; thus, gluing gives a global map $\mathcal{T} \rightarrow \mathcal{O}_X$, which induces a map $\mathcal{E} \rightarrow \Omega$. To see this is a locally split surjection, given a section $a \in \mathcal{L}$, we define a splitting $\sigma_a : \Omega \rightarrow \mathcal{E}$ by $\sigma_a(\omega)(b) = \langle \omega, b - a \rangle$. Now it is clear that the sequence in (iii) is exact, since the kernel of $\mathcal{E} \rightarrow \Omega$ comprises exactly the constant functions $\mathcal{L} \rightarrow \mathcal{O}_X$.

Now the injection $\mathcal{O}_X \rightarrow \mathcal{E}$ induces an injection $S^n(\mathcal{E}) \rightarrow S^{n+1}(\mathcal{E})$ for each $n \geq 0$; we now define $\mathcal{K} := \varinjlim S^n(\mathcal{E})$. Then \mathcal{K} has a natural filtration where $N_i \mathcal{K}$ is the image of $S^i \mathcal{E}$ in the direct limit. Also, an \mathcal{O}_X -algebra homomorphism $\mathcal{K} \rightarrow \mathcal{O}_X$ is equivalent to an \mathcal{O}_X -linear map $\mathcal{E} \rightarrow \mathcal{O}_X$ which maps the constant function with value 1 to 1; that is, a splitting of the above exact sequence on the left. The sheaf of such splittings forms a \mathcal{T} -torsor, where $D \in \mathcal{T}$ acts on the set of splittings by addition of the composition of $D : \Omega \rightarrow \mathcal{O}_X$ with the projection $\mathcal{E} \rightarrow \Omega$. Moreover, there is a morphism of torsors from \mathcal{L} to the torsor of left splittings given by $a \mapsto \epsilon_a$, where $\epsilon_a : \mathcal{E} \rightarrow \mathcal{O}_X$ is evaluation at a . This establishes the desired property of \mathcal{K} . This, in turn, easily implies that $\mathbf{Spec} \mathcal{K}$ represents the functor \mathcal{L} on a Zariski topos.

We now have a map $\mathcal{E} \rightarrow \mathcal{E} \otimes \Omega$ which sends $\phi \in \mathcal{E}$ to $\phi \otimes 1 + 1 \otimes \omega_\phi$, where ω_ϕ is the image of ϕ in Ω . Since a constant function a maps to $a \otimes 1$, this extends to a coaction map $\mathcal{K} \rightarrow \mathcal{K} \otimes S^* \Omega$. Similarly, a local splitting σ_a of the sequence in (iii) induces a map $1 \otimes \sigma_a - \sigma_a \otimes 1 : \Omega \rightarrow \mathcal{E} \otimes \mathcal{E}$, which is independent of a since the image of $\sigma_{a_2} - \sigma_{a_1}$ consists of constant functions. Thus, these local maps glue to a global map $\Omega \rightarrow \mathcal{E} \otimes \mathcal{E}$, which extends to a cosubtraction map $S^* \Omega \rightarrow \mathcal{K} \otimes \mathcal{K}$.

Finally, if X is a crystalline topos, then any extension of crystals is automatically another crystal. Thus, \mathcal{E} and therefore \mathcal{K} are also crystals. \square

3.2 Liftings of Frobenius as a Crystal

We consider the crystalline sites $\mathbf{Crys}(X/S)$ and $\mathbf{Crys}(X/\tilde{S})$. If (U, T) is an object of $\mathbf{Crys}(X/S)$, then since U is defined by a divided power ideal I_T in T , $a^p = 0$ for any $a \in I_T$. Therefore, the map $T \rightarrow T''$ factors through U'' . We thus get a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ T & \longrightarrow & U'' \end{array}$$

However, since the map $U' \rightarrow U''$ is étale, and $U \rightarrow T$ is an exact thickening, there exists a unique map $T \rightarrow U'$ making the above diagram commute. We let $f_{T/S} : T \rightarrow X'$ denote the composition of this map with the open embedding $U' \rightarrow X'$. Note that the pullback map $f_{T/S}^* : \Omega_{X'/S}^1 \rightarrow f_{T/S*} \Omega_{T/S}^1$ is zero. To see this, we may assume $X = U$; in this case, let $i' : U' \rightarrow T'$ be the closed immersion. Then after composing with the natural surjection $i'^* : i'^* \Omega_{T'/S}^1 \rightarrow \Omega_{U'/S}^1$, we get the map $f_{T/S}^* i'^* = F_{T/S}^* = 0$.

DEFINITION 3.3. Let \tilde{T} be an object of $\mathbf{Crys}(X/\tilde{S})$ which is flat over \tilde{S} , and let T be the closed subscheme defined by p .

- i) A *lifting of $f_{T/S}$ to \tilde{T}* is a lifting $\tilde{F} : \tilde{T} \rightarrow \tilde{X}'$ over \tilde{S} modulo p^2 . $\mathcal{L}_{X/S}(\tilde{T})$ is the set of all such liftings, and $\mathcal{L}_{X/S, \tilde{T}}$ is the sheaf on \tilde{T} of local liftings of $f_{T/S}$.

- ii) A *lifting* of $f_{T/S}$ is a pair (\tilde{T}', \tilde{F}) where \tilde{T}' is a flat object of $\text{Crys}(X/\tilde{S})$ and $\tilde{F} : \tilde{T}' \rightarrow \tilde{X}'$ is a lifting of $f_{T/S}$ to \tilde{T}' . An isomorphism $(\tilde{T}_1, \tilde{F}_1) \rightarrow (\tilde{T}_2, \tilde{F}_2)$ is a map $\tilde{g} : \tilde{T}_1 \rightarrow \tilde{T}_2$ which reduces to the identity on T and such that $\tilde{F}_1 = \tilde{F}_2 \circ \tilde{g}$. $\mathcal{L}_{\mathcal{X}/S, T}$ is the sheaf on T associated to the presheaf of isomorphism classes of local liftings of $f_{T/S}$.

Since \tilde{X}' is smooth over \tilde{S} , liftings of $f_{T/S}$ to \tilde{T} exist locally on T , so $\mathcal{L}_{\mathcal{X}/S, \tilde{T}}$ has nonempty stalks. A map $\tilde{g} : \tilde{T}_1 \rightarrow \tilde{T}_2$ induces a map $\mathcal{L}_{\mathcal{X}/S}(\tilde{g}) : \mathcal{L}_{\mathcal{X}/S}(\tilde{T}_2) \rightarrow \mathcal{L}_{\mathcal{X}/S}(\tilde{T}_1)$, defined by $\tilde{F} \mapsto \tilde{F} \circ \tilde{g}$. Now $\mathcal{L}_{\mathcal{X}/S, \tilde{T}}$ is a torsor over $\mathcal{H}om_{\mathcal{O}_T}(f_{T/S}^* \Omega_{\tilde{X}'/\tilde{S}}^1, \mathcal{O}_T) \simeq f_{T/S}^* \mathcal{T}_{\tilde{X}'/\tilde{S}}$. We will show that this torsor has a natural structure of a sheaf on $\text{Crys}(X/S)$, compatible with the crystal structure of the crystal $F_{X/S}^* \mathcal{T}_{X'/S}$, which maps $T \in \text{Crys}(X/S)$ to $f_{T/S}^* \mathcal{T}_{X'/S}$. (The corresponding connection is the Frobenius descent connection on $F_{X/S}^* \mathcal{T}_{X'/S}$.)

LEMMA 3.4. *Let \tilde{T}_1 and \tilde{T}_2 be two flat objects of $\text{Crys}(X/\tilde{S})$, and let $\tilde{g}_1, \tilde{g}_2 : \tilde{T}_1 \rightarrow \tilde{T}_2$ be two morphisms with the same reduction $g : T_1 \rightarrow T_2$ modulo p .*

- i) $\mathcal{L}_{\mathcal{X}/S}(\tilde{g}_1) = \mathcal{L}_{\mathcal{X}/S}(\tilde{g}_2)$ as maps $\mathcal{L}_{\mathcal{X}/S}(\tilde{T}_2) \rightarrow \mathcal{L}_{\mathcal{X}/S}(\tilde{T}_1)$.
 ii) For any flat object $\tilde{T} \in \text{Crys}(X/\tilde{S})$, the natural map $\mathcal{L}_{\mathcal{X}/S, \tilde{T}} \rightarrow \mathcal{L}_{\mathcal{X}/S, T}$ is an isomorphism, where T is the reduction of \tilde{T} modulo p .

Proof. For $\tilde{F} \in \mathcal{L}_{\mathcal{X}/S}(\tilde{T}_2)$, consider the commutative diagram

$$\begin{array}{ccccc} T_1 & \xrightarrow{\text{inc} \circ g} & \tilde{T}_2 & \xrightarrow{\tilde{F}} & \tilde{X}' \\ \downarrow & & \downarrow \Delta & & \downarrow \Delta' \\ \tilde{T}_1 & \xrightarrow{(\tilde{g}_1, \tilde{g}_2)} & \tilde{T}_2 \times_{\tilde{S}} \tilde{T}_2 & \xrightarrow{\tilde{F} \times \tilde{F}} & \tilde{X}' \times_{\tilde{S}} \tilde{X}' \end{array}$$

Then the left square induces the map $h : g^* \Omega_{\tilde{T}_2/S}^1 \rightarrow \mathcal{O}_{T_1}$ expressing the difference between \tilde{g}_1 and \tilde{g}_2 , and the right square induces the map $\tilde{F}^* : \tilde{F}^* \Omega_{\tilde{X}'/\tilde{S}}^1 \rightarrow \Omega_{\tilde{T}_2/\tilde{S}}^1$. Thus, overall the diagram induces the map $h \circ g^*(f_{T_2/S}^*) : g^* f_{T_2/S}^* \Omega_{\tilde{X}'/\tilde{S}}^1 \rightarrow \mathcal{O}_{T_1}$, which is zero since $f_{T_2/S}^* = 0$. This implies that $(\tilde{F} \circ \tilde{g}_1, \tilde{F} \circ \tilde{g}_2) : \tilde{T}_1 \rightarrow \tilde{X}' \times_{\tilde{S}} \tilde{X}'$ factors through Δ' , so $\tilde{F} \circ \tilde{g}_1 = \tilde{F} \circ \tilde{g}_2$.

To prove (ii) it suffices to prove the map is an isomorphism on stalks. Thus, for $t \in \tilde{T}$, if (\tilde{T}', \tilde{F}) is a lifting of $f_{T/S}$ on a neighborhood of t , then locally at t we have an isomorphism $\tilde{T} \simeq \tilde{T}'$, showing surjectivity. For injectivity, if $\tilde{F}, \tilde{F}' \in \mathcal{L}_{\mathcal{X}/S}(\tilde{T})$ become equal in $\mathcal{L}_{\mathcal{X}/S, T}$, then there is an automorphism of \tilde{T} which reduces to the identity modulo p and which carries \tilde{F} to \tilde{F}' . But then by (i), $\tilde{F} = \tilde{F}'$. \square

Hence given a map $g : T_1 \rightarrow T_2$, we can define a map $\theta_g : g^{-1} \mathcal{L}_{\mathcal{X}/S, T_2} \rightarrow \mathcal{L}_{\mathcal{X}/S, T_1}$ by gluing the maps $\mathcal{L}_{\mathcal{X}/S}(\tilde{g})$ for local liftings \tilde{g} of g . It is easy to see this map satisfies the cocycle condition $\theta_{hg} = \theta_h \circ h^{-1} \theta_g$. Also, since $\tilde{F}_2 \circ \tilde{g} - \tilde{F}_1 \circ \tilde{g} = g^*(\tilde{F}_2 - \tilde{F}_1) : g^* f_{T_2/S}^* \Omega_{\tilde{X}'/\tilde{S}}^1 \rightarrow \mathcal{O}_{T_1}$ for $\tilde{F}_i \in \mathcal{L}_{\mathcal{X}/S, T_2}$ if $\tilde{g} : \tilde{T}_1 \rightarrow \tilde{T}_2$ lifts g , θ_g respects the torsor structure.

Thus from (3.2) we get a crystal $\mathcal{K}_{\mathcal{X}/S}$ with filtration N , along with an extension

$$0 \rightarrow \mathcal{O}_{X/S} \rightarrow \mathcal{E}_{\mathcal{X}/S} \rightarrow F_{X/S}^* \Omega_{X'/S}^1 \rightarrow 0 \quad (5)$$

of crystals, where $\mathcal{E}_{\mathcal{X}/S} := N_1 \mathcal{K}_{\mathcal{X}/S}$.

We have an alternate construction of $\mathcal{E}_{\mathcal{X}/S}$, and hence $\mathcal{K}_{\mathcal{X}/S} \simeq \varinjlim S^n(\mathcal{E}_{\mathcal{X}/S})$, as follows: given an object $\tilde{T} \in \text{Crys}(X/\tilde{S})$ which is flat over \tilde{S} , we define $\mathcal{E}_{\mathcal{X}/S, \tilde{T}}$ to be the logarithmic conormal sheaf of the closed immersion

$$T \xrightarrow{\Gamma} T \times_S X' \xrightarrow{\text{inc}} \tilde{T} \times_{\tilde{S}} \tilde{X}',$$

where T is the reduction of \tilde{T} modulo p , and Γ is the graph of $f_{T/S} : T \rightarrow X'$. The functoriality properties of the conormal sheaf allow us to define maps $\theta_{\tilde{g}} : g^* \mathcal{E}_{\mathcal{X}/S, \tilde{T}_2} \rightarrow \mathcal{E}_{\mathcal{X}/S, \tilde{T}_1}$ for $\tilde{g} : \tilde{T}_1 \rightarrow \tilde{T}_2$, which it is possible to show depend only on the reduction $g : T_1 \rightarrow T_2$ of \tilde{g} modulo p . We then glue together $\mathcal{E}_{\mathcal{X}/S, \tilde{T}}$ to get the sheaves $\mathcal{E}_{\mathcal{X}/S, T}$. The exact sequence above comes from the exact sequence $\Gamma^* \check{\mathcal{N}}_{\tilde{T} \times \tilde{X}'/T \times X'} \rightarrow \check{\mathcal{N}}_{\tilde{T} \times \tilde{X}'/T} \rightarrow \check{\mathcal{N}}_{T \times X'/T} \rightarrow 0$. To connect this to the previous construction, observe that given $\tilde{F} \in \mathcal{L}_{\mathcal{X}/S, \tilde{T}}$, we can refactor the closed immersion above as

$$T \xrightarrow{\text{inc}} \tilde{T} \xrightarrow{\Gamma_{\tilde{F}}} \tilde{T} \times_{\tilde{S}} \tilde{X}'.$$

The corresponding map $\check{\mathcal{N}}_{\tilde{T} \times \tilde{X}'/T} \rightarrow \check{\mathcal{N}}_{\tilde{T}/T}$ gives a map $\mathcal{E}_{\mathcal{X}/S, \tilde{T}} \rightarrow \mathcal{O}_T$. Thus, for each element of $\mathcal{E}_{\mathcal{X}/S, \tilde{T}}$, we get a map $\mathcal{L}_{\mathcal{X}/S, \tilde{T}} \rightarrow \mathcal{O}_T$; it is then possible to show this map is in $N_1 \mathcal{K}_{\mathcal{X}/S, T}$, and that the induced map $\mathcal{E}_{\mathcal{X}/S, \tilde{T}} \rightarrow N_1 \mathcal{K}_{\mathcal{X}/S, T}$ is an isomorphism of extensions of $f_{T/S}^* \Omega_{X'/S}^1$ by \mathcal{O}_T .

3.3 Explicit Formulas: Connection and p -curvature

We now calculate the corresponding connection on $\mathcal{E}_{\mathcal{X}/S}$ and its p -curvature. The answer involves the following construction: suppose we have a lifting $\tilde{F} : \tilde{X} \rightarrow \tilde{X}'$ of $F_{X/S}$. Then since $F_{X/S}^* : \Omega_{X'/S}^1 \rightarrow F_{X/S*} \Omega_{X/S}^1$ is the zero map, $\tilde{F}^* : \Omega_{\tilde{X}'/\tilde{S}}^1 \rightarrow \tilde{F}_* \Omega_{\tilde{X}/\tilde{S}}^1$ is a multiple of p . Hence there is a unique map $\zeta_{\tilde{F}} : \Omega_{X'/S}^1 \rightarrow F_{X/S*} \Omega_{X/S}^1$ making the following diagram commute:

$$\begin{array}{ccc} \Omega_{\tilde{X}'/\tilde{S}}^1 & \xrightarrow{\tilde{F}^*} & \tilde{F}_* \Omega_{\tilde{X}/\tilde{S}}^1 \\ \downarrow & & \uparrow [p] \\ \Omega_{X'/S}^1 & \xrightarrow{\zeta_{\tilde{F}}} & F_{X/S*} \Omega_{X/S}^1. \end{array}$$

Now since (5) is an exact sequence of crystals, the only nontrivial part of the connection ∇ on $\mathcal{E}_{\mathcal{X}/S} \simeq \mathcal{O}_X \oplus F_{X/S}^* \Omega_{X'/S}^1$ is the part which maps $F_{X/S}^* \Omega_{X'/S}^1$ to $\mathcal{O}_X \otimes \Omega_{X/S}^1$, and similarly for the p -curvature $\psi(\nabla)$. (Note that the splitting $\mathcal{E}_{\mathcal{X}/S} \simeq \mathcal{O}_X \oplus F_{X/S}^* \Omega_{X'/S}^1$ depends on a lifting \tilde{F} of $F_{X/S}$.)

PROPOSITION 3.6. *Let \tilde{F} and $\zeta_{\tilde{F}}$ be as above.*

- i) (Mazur's Formula) $\zeta_{\tilde{F}}$ factors through $F_{X/S*} Z_{X/S}^1$ and induces a splitting of the exact sequence

$$0 \rightarrow F_{X/S*} B_{X/S}^1 \rightarrow F_{X/S*} Z_{X/S}^1 \xrightarrow{C_{X/S}} \Omega_{X'/S}^1 \rightarrow 0,$$

where $C_{X/S}$ is the Cartier operator.

- ii) The connection ∇ on $\mathcal{E}_{\mathcal{X}/S}$ satisfies

$$(\epsilon_{\tilde{F}} \otimes \text{id}) \circ \nabla \circ \sigma_{\tilde{F}} = -\zeta_{\tilde{F}} : F_{X/S}^* \Omega_{X'/S}^1 \rightarrow \Omega_{X/S}^1.$$

- iii) The p -curvature ψ of ∇ is equal to the composition

$$\mathcal{E}_{\mathcal{X}/S} \rightarrow F_{X/S}^* \Omega_{X'/S}^1 \xrightarrow{i \otimes \text{id}} \mathcal{E}_{\mathcal{X}/S} \otimes F_{X/S}^* \Omega_{X'/S}^1,$$

where $i : \mathcal{O}_X \rightarrow \mathcal{E}_{\mathcal{X}/S}$ is the inclusion map.

Proof. For $m \in \mathcal{M}_X^{\text{gp}}$, let $\tilde{m} \in \mathcal{M}_{\tilde{X}}^{\text{gp}}$ and $\tilde{m}' \in \mathcal{M}_{\tilde{X}'}^{\text{gp}}$ be liftings of m and $\pi^* m \in \mathcal{M}_{X'}^{\text{gp}}$, respectively. Then $\tilde{u} := \tilde{F}^*(\tilde{m}') - p\tilde{m}$ reduces to 0 in $\mathcal{M}_{X'}^{\text{gp}}$, so $\tilde{u} \in \mathcal{M}_{\tilde{X}}^{\text{gp}}$ and $\alpha_{\tilde{X}}(\tilde{u}) = 1 + [p]b$ for some $b \in \mathcal{O}_X$. Therefore,

$$[p]\zeta_{\tilde{F}}(d \log(\pi^* m)) = \tilde{F}^*(d \log(\tilde{m}')) = d \log(p\tilde{m}) + \frac{d\alpha(\tilde{u})}{\alpha(\tilde{u})} = [p] d \log(m) + [p]db.$$

Thus $\zeta_{\tilde{F}}(d\log(\pi^*m)) = d\log(m) + db \in Z_{X/S}^1$ and $C_{X/S}(\zeta_{\tilde{F}}(d\log(\pi^*m))) = C_{X/S}(d\log(m)) = d\log(\pi^*m)$ as required. Since $\{d\log(\pi^*m) : m \in \mathcal{M}_X^{\text{gp}}\}$ generates $\Omega_{X'/S}^1$ as an $\mathcal{O}_{X'}$ -module, this proves (i).

To prove (ii), let T be the first infinitesimal neighborhood of X in $X \times_S X$ and \tilde{T} the first infinitesimal neighborhood of \tilde{X} in $\tilde{X} \times_{\tilde{S}} \tilde{X}$, with its two natural projections $\tilde{p}_1, \tilde{p}_2 : \tilde{T} \rightarrow \tilde{X}$. Then for $\omega \in F_{X/S}^* \Omega_{X'/S}^1$, it is easy to see that $\theta_{p_i} \sigma_{\tilde{F}}(\omega) = \sigma_{\tilde{F} \circ \tilde{p}_i}(p_i^* \omega) = \sigma_{\tilde{F} \circ \tilde{p}_i}(1 \otimes \omega)$, so

$$\begin{aligned} (\epsilon_{\tilde{F}} \otimes \text{id}) \nabla \sigma_{\tilde{F}}(\omega) &= [\theta_{p_2} \sigma_{\tilde{F}}(\omega) - \theta_{p_1} \sigma_{\tilde{F}}(\omega)](\tilde{F} \circ \tilde{p}_1) = [\sigma_{\tilde{F} \circ \tilde{p}_2}(\omega) - \sigma_{\tilde{F} \circ \tilde{p}_1}(\omega)](\tilde{F} \circ \tilde{p}_1) \\ &= \langle \omega, \tilde{F} \circ \tilde{p}_1 - \tilde{F} \circ \tilde{p}_2 \rangle. \end{aligned}$$

However, by definition, the map $\tilde{F} \circ \tilde{p}_1 - \tilde{F} \circ \tilde{p}_2 : f_{T/S}^* \Omega_{X'/S}^1 \rightarrow \mathcal{O}_T$ is induced by pullback along the map $(\tilde{F} \circ \tilde{p}_2, \tilde{F} \circ \tilde{p}_1) : \tilde{T} \rightarrow \tilde{X}'_1$, where \tilde{X}'_1 is the first infinitesimal neighborhood (in the logarithmic sense) of the diagonal in $\tilde{X}' \times_{\tilde{S}} \tilde{X}'$. Now this map can be factored as the transposition map $(\tilde{p}_2, \tilde{p}_1) : \tilde{T} \rightarrow \tilde{T}$, which induces the map $-1 : \Omega_{\tilde{X}/\tilde{S}}^1 \rightarrow \Omega_{\tilde{X}/\tilde{S}}^1$, followed by the map $\tilde{F} \times \tilde{F} : \tilde{T} \rightarrow \tilde{X}'_1$, which induces the map $\tilde{F}^* : \tilde{F}^* \Omega_{\tilde{X}/\tilde{S}}^1 \rightarrow \Omega_{\tilde{X}/\tilde{S}}^1$. Therefore, $\tilde{F} \circ \tilde{p}_1 - \tilde{F} \circ \tilde{p}_2$ is the map induced by $-\tilde{F}^*$, which is exactly $-\zeta_{\tilde{F}}$.

Now for the p -curvature, since $\psi = 0$ on \mathcal{O}_X and on the quotient $F_{X/S}^* \Omega_{X'/S}^1$, ψ induces an endomorphism on $F_{X/S}^* \Omega_{X'/S}^1$; we need to show this endomorphism is the identity. It suffices to show that for $D' \in \mathcal{T}_{X'/S}$ and $\omega \in \Omega_{X'/S}^1$, we have $\psi_{D'}(\sigma_{\tilde{F}}(1 \otimes \omega)) = F_{X/S}^* \langle \omega, D' \rangle$. Furthermore, it suffices to consider the case in which $\omega = d\log(\pi^*m)$ for $m \in \mathcal{M}_X^{\text{gp}}$ and $D' = \pi^*D$ for $D = (\partial, \delta) \in \mathcal{T}_{X/S}$. Thus, suppose $\zeta_{\tilde{F}}(\omega) = d\log(m) + db$ for $b \in \mathcal{O}_X$. Then $\nabla_D(\sigma_{\tilde{F}}(1 \otimes \omega)) = -\delta m - \partial b \in \mathcal{O}_X$, so

$$\nabla_D^p(\sigma_{\tilde{F}}(1 \otimes \omega)) = -\partial^{p-1}(\delta m) - \partial^{(p)}b.$$

On the other hand, since $D^{(p)} = (\partial^{(p)}, \partial^{p-1} \circ \delta + F_X^* \circ \delta)$, we have

$$\nabla_{D^{(p)}}(\sigma_{\tilde{F}}(1 \otimes \omega)) = -\partial^{p-1}(\delta m) - F_X^*(\delta m) - \partial^{(p)}b.$$

Thus $\psi_{D'}(\sigma_{\tilde{F}}(1 \otimes d\log(m))) = F_X^*(\delta m) = F_{X/S}^* \langle \omega, D' \rangle$ as required. \square

PROPOSITION 3.7. *Let $\psi : \mathcal{K}_{X/S} \rightarrow \mathcal{K}_{X/S} \otimes F_{X/S}^* \Omega_{X'/S}^1$ be the p -curvature of the canonical connection on $\mathcal{K}_{X/S}$. Then there is a canonical commutative diagram*

$$\begin{array}{ccc} \mathcal{K}_{X/S} & \xrightarrow{\psi} & \mathcal{K}_{X/S} \otimes F_{X/S}^* \Omega_{X'/S}^1 \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbf{T}(\mathcal{L})} & \xrightarrow{d} & \Omega_{\mathbf{T}(\mathcal{L})/X}^1, \end{array}$$

in which the vertical arrows are isomorphisms. (Here we give $\mathbf{T}(\mathcal{L}_{X/S})$ the log structure induced by that of X .)

Proof. Since $\mathcal{K}_{X/S}$ is a crystal of $\mathcal{O}_{X/S}$ -algebras, the corresponding connection on $\mathcal{K}_{X/S}$ satisfies the Leibniz rule, so its p -curvature does also. Hence ψ is a derivation which annihilates \mathcal{O}_X ; it will suffice to show it is the universal derivation on $\mathcal{K}_{X/S}$ over \mathcal{O}_X . Given any derivation $D : \mathcal{K}_{X/S} \rightarrow \mathcal{F}$ over X , the restriction of D to $\mathcal{E}_{X/S}$ factors through the quotient map to $F_{X/S}^* \Omega_{X'/S}^1$. The \mathcal{O}_X -linear map $F_{X/S}^* \Omega_{X'/S}^1 \rightarrow \mathcal{F}$ induces a $\mathcal{K}_{X/S}$ -linear map $\mathcal{K}_{X/S} \otimes F_{X/S}^* \Omega_{X'/S}^1 \rightarrow \mathcal{F}$. Using the formula for ψ from (3.6), we easily see that the composition of this map with ψ agrees with D on $\mathcal{E}_{X/S}$. Since $\mathcal{E}_{X/S}$ generates $\mathcal{K}_{X/S}$ as a ring and both D and ψ satisfy the Leibniz rule, this shows that D factors through ψ . \square

Global liftings of $F_{X/S}$ rarely exist; this limits the effectiveness of generating elements of $\mathcal{E}_{X/S}$ using $\sigma_{\tilde{F}}$ and the inclusion of \mathcal{O}_X . On the other hand, global liftings of $\pi_{X/S} : X' \rightarrow X$ exist more often. For example, in the situation in which the lifting $\tilde{X}' \rightarrow \tilde{S}$ comes from a lifting $F_{\tilde{S}}$ of F_S , we can define $\tilde{\pi}$ to be the map $\tilde{X}' \rightarrow \tilde{X}''$ composed with the projection $\tilde{X}'' = \tilde{X} \times_{F_{\tilde{S}}} \tilde{S} \rightarrow \tilde{X}$. The following result gives an alternate way to generate elements of $\mathcal{E}_{X/S}$ using $\tilde{\pi}$ instead of \tilde{F} .

PROPOSITION 3.8. *Suppose that $\tilde{\pi} : \tilde{X}' \rightarrow \tilde{X}$ lifts $\pi_{X/S} : X' \rightarrow X$. Then there is a unique map $\beta_{\tilde{\pi}}(\tilde{m}) : \mathcal{M}_{\tilde{X}}^{\text{gp}} \rightarrow \mathcal{E}_{X/S}$ such that for each section $\tilde{m} \in \mathcal{M}_{\tilde{X}}^{\text{gp}}$ and each local lifting \tilde{F} of $F_{X/S}$, we have*

$$1 + [p]\beta_{\tilde{\pi}}(\tilde{m})(\tilde{F}) = \alpha_{\tilde{X}}(\tilde{F}^* \tilde{\pi}^* \tilde{m} - p\tilde{m}).$$

Furthermore:

i) $\beta_{\tilde{\pi}}$ induces a surjective map $\mathcal{M}_{\tilde{X}}^{\text{gp}} \otimes_{\mathbb{Z}} \mathcal{O}_X \rightarrow \mathcal{E}_{X/S}$.

ii) We have

$$[p]\alpha(m)^p \beta_{\tilde{\pi}}(\tilde{m})(\tilde{F}) = \tilde{F}^* \tilde{\pi}^* \alpha(\tilde{m}) - \alpha(\tilde{m})^p.$$

(Thus if we define $\delta_{\tilde{\pi}} : \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{E}_{X/S}$ so that $[p]\delta_{\tilde{\pi}}(\tilde{a})(\tilde{F}) = \tilde{F}^* \tilde{\pi}^* \tilde{a} - \tilde{a}^p$, then

$$\alpha(m)^p \beta_{\tilde{\pi}}(\tilde{m}) = \delta_{\tilde{\pi}}(\alpha(\tilde{m})).$$

This map $\delta_{\tilde{\pi}}$ matches the construction of $\delta_{\tilde{\pi}}$ in the nonlogarithmic case [OV].)

iii) The following diagram commutes:

$$\begin{array}{ccccc} \mathcal{O}_X & \xrightarrow{\alpha_{\tilde{X}}^{-1}(1+[p]\cdot)} & \mathcal{M}_{\tilde{X}}^{\text{gp}} & \longrightarrow & \mathcal{M}_{\tilde{X}}^{\text{gp}} \\ \downarrow F_X^* & & \downarrow \beta_{\tilde{\pi}} & & \downarrow 1 \otimes (d \log \circ \pi^*) \\ \mathcal{O}_X & \longrightarrow & \mathcal{E}_{X/S} & \longrightarrow & F_{X/S}^* \Omega_{X'/S}^1. \end{array}$$

iv) If $\tilde{F} : \tilde{X} \rightarrow \tilde{X}'$ lifts $F_{X/S}$, then for each $\tilde{m} \in \mathcal{M}_{\tilde{X}}^{\text{gp}}$,

$$\beta_{\tilde{\pi}}(\tilde{m}) = \beta_{\tilde{\pi}}(\tilde{m})(\tilde{F}) + \sigma_{\tilde{F}}(1 \otimes d \log(\pi^* m)),$$

where $m \in \mathcal{M}_X^{\text{gp}}$ is the reduction of \tilde{m} .

v) For every local section \tilde{m} of $\mathcal{M}_{\tilde{X}}^{\text{gp}}$ lifting some $m \in \mathcal{M}_X^{\text{gp}}$,

$$\begin{aligned} \nabla \beta_{\tilde{\pi}}(\tilde{m}) &= -1 \otimes d \log m, \\ \psi \beta_{\tilde{\pi}}(\tilde{m}) &= 1 \otimes (1 \otimes d \log(\pi^* m)). \end{aligned}$$

Proof. The pullback of $\tilde{F}^* \tilde{\pi}^* \tilde{m} - p\tilde{m}$ to X is $F_{X/S}^* \pi^* m - pm = 0$, which implies $\tilde{F}^* \tilde{\pi}^* \tilde{m} - p\tilde{m} \in \mathcal{M}_{\tilde{X}}^*$ since the inclusion $X \rightarrow \tilde{X}$ is exact, and $\alpha_{\tilde{X}}(\tilde{F}^* \tilde{\pi}^* \tilde{m} - p\tilde{m})$ pulls back to 1 in X and is therefore in $1 + [p]\mathcal{O}_X$. Now for $\tilde{F}_1, \tilde{F}_2 \in \mathcal{L}_{X/S}$, we have

$$\alpha_{\tilde{X}}(\tilde{F}_2^* \tilde{\pi}^* \tilde{m} - \tilde{F}_1^* \tilde{\pi}^* \tilde{m}) = 1 + [p]\langle d \log(\pi^* m), \tilde{F}_2 - \tilde{F}_1 \rangle.$$

Multiplying this by the equation $\alpha_{\tilde{X}}(\tilde{F}_1^* \tilde{\pi}^* \tilde{m} - p\tilde{m}) = 1 + [p]\beta_{\tilde{\pi}}(\tilde{m})(\tilde{F}_1)$, we get

$$\alpha_{\tilde{X}}(\tilde{F}_2^* \tilde{\pi}^* \tilde{m} - \tilde{F}_1^* \tilde{\pi}^* \tilde{m}) = 1 + [p](\beta_{\tilde{\pi}}(\tilde{m})(\tilde{F}_1) + \langle d \log(\pi^* m), \tilde{F}_2 - \tilde{F}_1 \rangle) = 1 + [p]\beta_{\tilde{\pi}}(\tilde{m})(\tilde{F}_2).$$

Hence $\beta_{\tilde{\pi}}(\tilde{m})(\tilde{F}_2) - \beta_{\tilde{\pi}}(\tilde{m})(\tilde{F}_1) = \langle \pi^*(d \log m), \tilde{F}_2 - \tilde{F}_1 \rangle$; this proves that $\beta_{\tilde{\pi}}(\tilde{m}) \in \mathcal{E}_{X/S}$ and also shows the commutativity of the right square in (iii).

Now multiplying both sides of the equation defining $\beta_{\tilde{\pi}}(\tilde{m})$ by $\alpha(p\tilde{m})$, we get

$$\alpha_{\tilde{X}}(\tilde{m})^p + [p]\alpha(m)^p \beta_{\tilde{\pi}}(\tilde{F}) = \alpha_{\tilde{X}}(\tilde{F}^* \tilde{\pi}^* \tilde{m}).$$

Subtracting $\alpha_{\tilde{X}}(\tilde{m})^p$ from both sides gives (ii). In particular, if $\alpha(\tilde{m}) = 1 + [p]a$, then $\alpha(\tilde{m})^p = 1$, while $\tilde{F}^*\tilde{\pi}^*\alpha(\tilde{m}) = 1 + [p]F_X^*a$, so (ii) reduces to $[p]\beta_{\tilde{\pi}}(\tilde{m}) = [p]F_X^*a$, which shows the commutativity of the left square in (iii).

To prove (i), we first show $\beta_{\tilde{\pi}}$ is additive. To see this, we multiply the equations defining $\beta_{\tilde{\pi}}(\tilde{m}_1)$ and $\beta_{\tilde{\pi}}(\tilde{m}_2)$ to get

$$1 + [p](\beta_{\tilde{\pi}}(\tilde{m}_1) + \beta_{\tilde{\pi}}(\tilde{m}_2))(\tilde{F}) = \alpha_{\tilde{X}}(\tilde{F}^*\tilde{\pi}^*(\tilde{m}_1 + \tilde{m}_2) - p(\tilde{m}_1 + \tilde{m}_2)) = 1 + [p]\beta_{\tilde{\pi}}(\tilde{m}_1 + \tilde{m}_2)(\tilde{F}).$$

Combining this fact with (iii) easily proves (i).

The formula for the p -curvature in (v) follows directly from (iii). Also, it suffices to verify the formula for $\nabla\beta_{\tilde{\pi}}(\tilde{m})$ locally, so we may assume we have a lifting \tilde{F} of $F_{X/S}$. Let $g_{\tilde{m}} := \beta_{\tilde{\pi}}(\tilde{m})(\tilde{F})$; then from (iii) we conclude $\nabla\beta_{\tilde{\pi}}(\tilde{m}) = 1 \otimes dg_{\tilde{m}} - 1 \otimes \zeta_{\tilde{F}}(d\log(\pi^*m))$. On the other hand,

$$\begin{aligned} \tilde{F}^*(d\log(\tilde{\pi}^*\tilde{m})) &= [p]d\log(m) + d\log(\tilde{F}^*\tilde{\pi}^*\tilde{m} - p\tilde{m}) \\ &= [p]d\log(m) + \frac{d(1 + [p]g_{\tilde{m}})}{1 + [p]g_{\tilde{m}}} = [p]d\log(m) + [p]dg_{\tilde{m}}. \end{aligned}$$

Hence $\zeta_{\tilde{F}}(d\log(\pi^*m)) = d\log(m) + dg_{\tilde{m}}$, and $\nabla\beta_{\tilde{\pi}}(\tilde{m}) = -1 \otimes d\log(m)$ as required. \square

In terms of the construction of $\mathcal{E}_{X/S}$ as a conormal sheaf, we can characterize $\beta_{\tilde{\pi}}$ as follows: for $\tilde{m} \in \mathcal{M}_{\tilde{X}}^{\text{gp}}$, the pullback of $(-p\tilde{m}, \tilde{\pi}^*\tilde{m}) \in \mathcal{M}_{\tilde{X} \times \tilde{X}'}^{\text{gp}}$ to $\mathcal{M}_{\tilde{X}}^{\text{gp}}$ is 0, which implies that it becomes a unit in the first infinitesimal neighborhood X_1 of X in $\tilde{X} \times_{\tilde{S}} \tilde{X}'$. Then $\beta_{\tilde{\pi}}(\tilde{m}) = \alpha_{X_1}(-p\tilde{m}, \tilde{\pi}^*\tilde{m}) - 1$.

3.4 Functoriality

The geometric construction of $\mathcal{E}_{X/S}$ given above makes it straightforward to check its functoriality. First we treat the general functoriality properties of $\mathbf{T}(\mathcal{L})$.

LEMMA 3.9. *Suppose we have a morphism $h : X \rightarrow Y$ of ringed topoi.*

- i) *Let $\mathcal{T}_1 \rightarrow \mathcal{T}_2$ be a map of locally free sheaves on X , and let $\Omega_i := \tilde{\mathcal{T}}_i$. Then for \mathcal{L}_1 a \mathcal{T}_1 -torsor, let $\mathcal{L}_2 := \mathcal{L}_1 \wedge_{\mathcal{T}_1} \mathcal{T}_2$ be the \mathcal{T}_2 -torsor induced from \mathcal{L}_1 . (Note that this is the unique \mathcal{T}_2 -torsor \mathcal{L}_2 with a map $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ respecting the actions.) Then letting \mathcal{K}_i be the \mathcal{O}_X -algebra representing \mathcal{L}_i , we have a natural map $\mathcal{K}_2 \rightarrow \mathcal{K}_1$ respecting the filtration N , and the restriction to $\mathcal{E}_2 \rightarrow \mathcal{E}_1$ fits into a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \Omega_2 \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \Omega_1 \longrightarrow 0. \end{array}$$

- ii) *Let \mathcal{T} be a locally free sheaf on Y , and let \mathcal{L} be an \mathcal{T} -torsor on Y . Then we have a natural isomorphism $h^*\mathcal{K}_{\mathcal{L}} \simeq \mathcal{K}_{h^*\mathcal{L}}$ respecting the filtration N , where $h^*\mathcal{L} := h^{-1}\mathcal{L} \wedge_{h^{-1}\mathcal{T}} h^*\mathcal{T}$ is the induced $h^*\mathcal{T}$ -torsor. Furthermore, the restriction $h^*\mathcal{E}_{\mathcal{L}} \simeq \mathcal{E}_{h^*\mathcal{L}}$ is an isomorphism of extensions of $h^*\Omega$ by \mathcal{O}_X .*

Proof. For (i), the map $\mathcal{E}_2 \rightarrow \mathcal{E}_1$ is induced by composition with the map $\mathcal{L}_1 \rightarrow \mathcal{L}_2$. Since the map $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ respects the actions of \mathcal{T}_1 and \mathcal{T}_2 , we see that the image of \mathcal{E}_2 is in fact contained in \mathcal{E}_1 , and we also get the commutativity of the right square in the diagram above. The commutativity of the left square is obvious. This induces the map $\mathcal{K}_2 \rightarrow \mathcal{K}_1$.

For (ii), we have a map from $h^{-1}\mathcal{L}$ to the $h^*\mathcal{T}$ -torsor of splittings of

$$0 \rightarrow h^*\mathcal{O}_Y \rightarrow h^*\mathcal{E}_{\mathcal{L}} \rightarrow h^*\Omega \rightarrow 0$$

on the left, which for $a \in \mathcal{L}$ sends $h^{-1}a \in h^{-1}\mathcal{L}$ to $h^*\epsilon_a$, where ϵ_a is evaluation at a . This induces a morphism of $h^*\mathcal{T}$ -torsors from $h^*\mathcal{L}$ to the torsor of splittings. However, the torsor of splittings

determines an extension uniquely; to see this, given an extension $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \Omega \rightarrow 0$, the dual is an extension

$$0 \rightarrow \mathcal{T} \rightarrow \check{\mathcal{E}} \rightarrow \mathcal{O}_X \rightarrow 0.$$

Then the torsor of splittings is just the inverse image of $1 \in \mathcal{O}_X$ in $\check{\mathcal{E}}$; however, taking the inverse image of 1 is exactly the canonical isomorphism $\text{Ext}^1(\mathcal{O}_X, \mathcal{T}) \simeq H^1(X, \mathcal{T})$. Therefore, we get an isomorphism of extensions of $h^*\Omega$ by \mathcal{O}_X , $h^*\mathcal{E}_{\mathcal{L}} \simeq \mathcal{E}_{h^*\mathcal{L}}$. Again, this easily extends to an isomorphism $h^*\mathcal{K}_{\mathcal{L}} \simeq \mathcal{K}_{h^*\mathcal{L}}$. \square

By a morphism $\mathcal{X}/S \rightarrow \mathcal{Y}/S$ we mean a map $h : X \rightarrow Y$ along with a lifting $\tilde{h}' : \tilde{X}' \rightarrow \tilde{Y}'$ of h' .

PROPOSITION 3.10. *A morphism $(h, \tilde{h}') : \mathcal{X}/S \rightarrow \mathcal{Y}/S$, where X and Y are smooth S -schemes, induces a horizontal morphism $\theta_{h, \tilde{h}'} : h^*\mathcal{K}_{\mathcal{Y}/S} \rightarrow \mathcal{K}_{\mathcal{X}/S}$ respecting the filtration N ., such that the restriction to $h^*\mathcal{E}_{\mathcal{Y}/S} \rightarrow \mathcal{E}_{\mathcal{X}/S}$ fits into a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & h^*\mathcal{O}_Y & \longrightarrow & h^*\mathcal{E}_{\mathcal{Y}/S} & \longrightarrow & h^*F_{Y/S}^*\Omega_{Y'/S}^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow \theta_{h, \tilde{h}'} & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{E}_{\mathcal{X}/S} & \longrightarrow & F_{X/S}^*\Omega_{X'/S}^1 \longrightarrow 0. \end{array}$$

Here the map $h^*\mathcal{O}_Y \rightarrow \mathcal{O}_X$ is the standard isomorphism, and the map $h^*F_{Y/S}^*\Omega_{Y'/S}^1 \simeq F_{X/S}^*h'^*\Omega_{Y'/S}^1 \rightarrow F_{X/S}^*\Omega_{X'/S}^1$ is the pullback by $F_{X/S}^*$ of the natural map $h'^*\Omega_{Y'/S}^1 \rightarrow \Omega_{X'/S}^1$.

For fixed h , let $\tilde{h}'_1, \tilde{h}'_2$ be two liftings of h' , and let $D : h'^*\Omega_{Y'/S}^1 \rightarrow \mathcal{O}_{X'}$ express their difference. Then $\theta_{h, \tilde{h}'_2} - \theta_{h, \tilde{h}'_1} : h^*\mathcal{E}_{\mathcal{Y}/S} \rightarrow \mathcal{E}_{\mathcal{X}/S}$ is the composition of the projection $h^*\mathcal{E}_{\mathcal{Y}/S} \rightarrow h^*F_{Y/S}^*\Omega_{Y'/S}^1$, the map

$$F_{X/S}^*(D) : h^*F_{Y/S}^*\Omega_{Y'/S}^1 \simeq F_{X/S}^*h'^*\Omega_{Y'/S}^1 \rightarrow \mathcal{O}_X,$$

and the inclusion $\mathcal{O}_X \rightarrow \mathcal{E}_{\mathcal{X}/S}$.

Proof. Let T_1, T_2 be objects of $\text{Crys}(X/S)$, $\text{Crys}(Y/S)$, respectively, and let $g : T_1 \rightarrow T_2$ be a PD morphism extending $h : X \rightarrow Y$. (Thus g is a section of $h^{-1}T_2(T_1)$.) Then as in the construction of the crystalline structure of $\mathcal{L}_{\mathcal{X}/S}$, composing with liftings \tilde{g} of g induces a morphism of $h^*F_{Y/S}^*\mathcal{T}_{Y'/S}$ -torsors $g^*\mathcal{L}_{\mathcal{Y}/S, T_2} \rightarrow \mathcal{L}_{\mathcal{Y}/S, X, T_1}$, where $\mathcal{L}_{\mathcal{Y}/S, X, T_1}$ denotes the sheaf of liftings of $f_{T_2/S} \circ g : X \rightarrow Y'$. This is clearly compatible with the transition maps for $\mathcal{L}_{\mathcal{Y}/S, X}$ and $\mathcal{L}_{\mathcal{Y}/S}$, so we get a morphism of sheaves $h^*\mathcal{L}_{\mathcal{Y}/S} \rightarrow \mathcal{L}_{\mathcal{Y}/S, X}$ on the crystalline site $\text{Crys}(X/S)$. On the other hand, composing a lifting $\tilde{T}_1 \rightarrow \tilde{X}'$ with \tilde{h}' gives a morphism of torsors $\mathcal{L}_{\mathcal{X}/S} \rightarrow \mathcal{L}_{\mathcal{Y}/S, X}$. Thus, overall, h and \tilde{h}' induce a morphism of torsors $\mathcal{L}_{\mathcal{X}/S} \wedge_{\mathcal{T}_1} \mathcal{T}_2 \rightarrow h^*\mathcal{L}_{\mathcal{Y}/S}$, which induces the desired map $\theta_{h, \tilde{h}'}$.

Now given $\tilde{h}'_1, \tilde{h}'_2 : \tilde{X}' \rightarrow \tilde{Y}'$ with difference $D : h'^*\Omega_{Y'/S}^1 \rightarrow \mathcal{O}_{X'}$, for $\tilde{F} \in \mathcal{L}_{\mathcal{X}/S}$, we have $\tilde{h}'_2 \circ \tilde{F} - \tilde{h}'_1 \circ \tilde{F} = F_{X/S}^*(D) : F_{X/S}^*h'^*\Omega_{Y'/S}^1 \rightarrow \mathcal{O}_X$. Thus, for $\phi \in \mathcal{E}_{\mathcal{Y}/S}$ with image $\omega \in F_{Y/S}^*\Omega_{Y'/S}^1$,

$$(\theta_{h, \tilde{h}'_2}\phi - \theta_{h, \tilde{h}'_1}\phi)(\tilde{F}) = (\theta_h\phi)(\tilde{h}'_2 \circ \tilde{F}) - (\theta_h\phi)(\tilde{h}'_1 \circ \tilde{F}) = \langle h^*\omega, \tilde{h}'_2 \circ \tilde{F} - \tilde{h}'_1 \circ \tilde{F} \rangle,$$

which implies that $\theta_{h, \tilde{h}'_2}\phi - \theta_{h, \tilde{h}'_1}\phi$ is the constant $(F_{X/S}^*D)(\omega)$. \square

COROLLARY 3.11. *A morphism $(h, \tilde{h}') : \mathcal{X}/S \rightarrow \mathcal{Y}/S$, where X and Y are smooth S -schemes, induces an exact sequence*

$$h^*\mathcal{E}_{\mathcal{Y}/S} \rightarrow \mathcal{E}_{\mathcal{X}/S} \rightarrow F_{X/S}^*\Omega_{X'/Y'}^1 \rightarrow 0.$$

If h is smooth, this sequence is short exact and locally split. Similarly, if h is a closed immersion and $F_{X/S}$ is flat, there is an exact and locally split sequence

$$0 \rightarrow F_{X/S}^*\check{\mathcal{N}}_{Y'/X'} \rightarrow h^*\mathcal{E}_{\mathcal{Y}/S} \rightarrow \mathcal{E}_{\mathcal{X}/S} \rightarrow 0.$$

4. The Cartier Transform

4.1 PD Higgs Fields and p -crystals

Suppose we are given a lifting \tilde{X}'/\tilde{S} of X'/S ; we shall see that $\mathcal{X}/\mathcal{S} := (X/S, \tilde{X}'/\tilde{S})$ determines a splitting of the Azumaya algebra $\tilde{\mathcal{D}}_{X/S}$ over $\mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{\mathcal{G}}$, where \mathcal{G} is the nilpotent divided power envelope of the zero section of the cotangent bundle of X'/S , so that $\mathcal{O}_{\mathcal{G}} = \hat{\Gamma} \cdot \mathcal{T}_{X'/S}$. We thus let $\mathcal{O}_{\mathcal{G}}^{\mathcal{B}} := \mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{\mathcal{G}} \simeq \tilde{\mathfrak{Z}} \otimes_{S \cdot \mathcal{T}_{X'/S}} \mathcal{O}_{\mathcal{G}}$, where $\tilde{\mathfrak{Z}}$ is the center of $\tilde{\mathcal{D}}_{X/S}$. We also let $\mathcal{O}_{\mathcal{G}}^{\mathcal{A}} := \mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{\mathcal{G}} \simeq \mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{B}_{X/S}} \mathcal{O}_{\mathcal{G}}^{\mathcal{B}}$, and $\tilde{\mathcal{D}}_{X/S}^{\gamma} := \tilde{\mathcal{D}}_{X/S} \otimes_{\tilde{\mathfrak{Z}}} \mathcal{O}_{\mathcal{G}}^{\mathcal{B}} \simeq \tilde{\mathcal{D}}_{X/S} \otimes_{S \cdot \mathcal{T}_{X'/S}} \mathcal{O}_{\mathcal{G}}$.

Fix an $\overline{\mathcal{M}}_X^{\text{gp}}$ -set \mathcal{J} . We denote by $HIG_{PD}^{\mathcal{B}}(X'/S)$ the category of \mathcal{J} -indexed $\mathcal{O}_{\mathcal{G}}^{\mathcal{B}}$ -modules, or equivalently the category of \mathcal{J} -indexed $\mathcal{B}_{X/S}$ -modules E' equipped with a $\mathcal{B}_{X/S}$ -linear \mathcal{G} -Higgs field

$$\theta : \mathcal{O}_{\mathcal{G}} \rightarrow \mathcal{E}nd_{\mathcal{B}_{X/S}}(E').$$

Similarly, we denote by $MIC_{PD}^{\mathcal{A}}(X/S)$ the category of \mathcal{J} -indexed $\tilde{\mathcal{D}}_{X/S}^{\gamma}$ -modules. An object of $MIC_{PD}^{\mathcal{A}}(X/S)$ is equivalent to a \mathcal{J} -indexed $\mathcal{A}_X^{\text{gp}}$ -module E with an integrable and admissible connection ∇ , along with an $\mathcal{A}_X^{\text{gp}}$ -linear \mathcal{G} -Higgs field

$$\theta : \mathcal{O}_{\mathcal{G}} \rightarrow F_{X/S*} \mathcal{E}nd_{\mathcal{A}_X^{\text{gp}}}(E)$$

extending the F -Higgs field

$$\psi : S \cdot \mathcal{T}_{X'/S} \rightarrow F_{X/S*} \mathcal{E}nd_{\mathcal{A}_X^{\text{gp}}}(E)$$

given by the p -curvature of ∇ . We also let $MIC_{PD}^{\mathcal{A}}(X/S)$ and $HIG_{PD}^{\mathcal{B}}(X'/S)$ be the full subcategories of locally nilpotent objects of $MIC_{PD}^{\mathcal{A}}(X/S)$ and $HIG_{PD}^{\mathcal{B}}(X'/S)$, respectively, that is, the objects such that every element of E or E' is annihilated by a sufficiently high power of $\Gamma_+ \mathcal{T}_{X'/S}$.

For example, let $MIC_{\ell}^{\mathcal{A}}(X/S)$ be the category of $\mathcal{A}_X^{\text{gp}}$ -modules with integrable, admissible connection ∇ such that the p -curvature ψ is nilpotent of level $\leq \ell$; then for $\ell < p$ we may consider $MIC_{\ell}^{\mathcal{A}}(X/S)$ as a full subcategory of $MIC_{PD}^{\mathcal{A}}(X/S)$ by letting $\Gamma_{\geq p} \mathcal{T}_{X'/S}$ act as zero. Now the convolution product coming from the group scheme structure of \mathcal{G} allows us to make $MIC_{PD}^{\mathcal{A}}(X/S)$ and $HIG_{PD}^{\mathcal{B}}(X'/S)$ into tensor categories; in particular, the total \mathcal{G} -Higgs field on a tensor product is given by

$$\theta_{(D')^{[n]}} = \sum_{i+j=n} \theta_{(D')^{[i]}} \otimes \theta_{(D')^{[j]}}.$$

Note that, in particular, $\theta_{(D')^{[p]}}$ can be nonzero on $E_1 \otimes E_2$ even if $E_1, E_2 \in MIC_{\ell}^{\mathcal{A}}(X/S)$ for $\ell < p$.

We shall see that the Azumaya algebra $\tilde{\mathcal{D}}_{X/S}$ splits over $\mathcal{O}_{\mathcal{G}}^{\mathcal{B}}$, which gives a Riemann-Hilbert correspondence between $MIC_{PD}^{\mathcal{A}}(X/S)$ and $HIG_{PD}^{\mathcal{B}}(X'/S)$. We shall also use this splitting to get a Riemann-Hilbert correspondence for certain \mathcal{O}_X -modules with connection. For this, let $\mathcal{D}_{X/S}^{\gamma} := \mathcal{D}_{X/S} \otimes_{S \cdot \mathcal{T}_{X'/S}} \mathcal{O}_{\mathcal{G}}$, and let $MIC_{PD}(X/S)$ and $HIG_{PD}(X'/S)$ be the categories of $\mathcal{D}_{X/S}^{\gamma}$ - and $\mathcal{O}_{\mathcal{G}}$ -modules, respectively. We also let $MIC_{PD}(X/S)$ and $HIG_{PD}(X'/S)$ be the corresponding full subcategories of locally nilpotent objects.

We can provide interpretations of the categories $MIC_{PD}(X/S)$ and $MIC_{PD}^{\mathcal{A}}(X/S)$ which are crystalline in nature. Our motivation for the following theory is the fact that the ring of differential operators on X over S is very similar to $\Gamma \cdot \mathcal{T}_{X/S}$, except for the fact that it is noncommutative. Thus, in our theory we will allow p th divided powers of elements of the ideal of the diagonal in $X_S X$, in order to preserve the action of the image of $c' : F_{X/S}^* \mathcal{T}_{X'/S} \rightarrow \mathcal{D}_{X/S}$, but no higher divided powers.

In particular, define a p -PD ring A to be an algebra over a given ring B of characteristic p , with

an ideal I of B such that $I^{(p)} := \{x^p : x \in I\} = (0)$, and a map $\gamma_p : I \rightarrow A$ satisfying

$$\gamma_p(x+y) = \gamma_p(x) + \gamma_p(y) + \sum_{i=1}^{p-1} \frac{x^{p-i}y^i}{(p-i)!i!},$$

$$\gamma_p(\lambda x) = \lambda^p \gamma_p(x).$$

As usual, we write $x^{[p]} := \gamma_p(x)$. Similarly to the case of full PD ideals, we can show that there is a left adjoint $\Gamma_{R,(p)}$ to the functor $(A, B, I, \gamma) \mapsto I$ from the category of p -PD rings over R to the category of R -modules. Similarly, given a closed immersion $Y \rightarrow X$, there is a right universal p -PD scheme $D_{Y,(p)}$ with an exact closed immersion $Y \rightarrow D_{Y,0,(p)}$ defined by the p -PD ideal of $D_{Y,(p)}$ and a map $D_{Y,0,(p)} \rightarrow X$; we call $D_{Y,(p)}$ the p -PD envelope of Y . (Here $D_{Y,(p)}$ corresponds to A , and $D_{Y,0,(p)}$ corresponds to B .) We now define $\text{Crys}_{(p)}(X/S)$ as the category of tuples $(U, T, V, i, f, \gamma_p)$, where U is an open subscheme of X , $f : V \rightarrow T$ is a strict affine map, and $i : U \rightarrow T$ is an exact closed immersion defined by a p -PD ideal $(I_T, \gamma_p : I_T \rightarrow f_* \mathcal{O}_V)$. For example, for each n we may define $D_{(p)}(n) \in \text{Crys}_{(p)}(X/S)$ to be the p -PD envelope of the diagonal embedding $\Delta : X \rightarrow X^{n+1}$. Also, for $(U, T, i, \gamma) \in \text{Crys}(X/S)$, we have an object $(U, T, T, i, \text{id}, \gamma_p) \in \text{Crys}_{(p)}(X/S)$, so that we may regard $\text{Crys}(X/S)$ as a (full) subcategory of $\text{Crys}_{(p)}(X/S)$. We let $\mathcal{O}_{X/S}$ be the sheaf of rings on $\text{Crys}_{(p)}(X/S)$ which to $(U, T, V, i, f, \gamma_p)$ associates \mathcal{O}_V .

Locally on X , if we have a logarithmic system of coordinates m_1, \dots, m_r , then as usual define $\eta_i := \alpha_{D_{0,(p)}(1)}(-m_i, m_i) - 1$. Then a basis for $\mathcal{D}_{(p)}(1) := \mathcal{O}_{D_{0,(p)}(1)}$ as an \mathcal{O}_X -module is $\{\eta^I (\eta^{[p]})^J : I \in M := \{0, \dots, p-1\}^r, J \in \mathbb{N}^r\}$, and $\mathcal{O}_{D_{0,(p)}(1)}$ has basis $\{\eta^I : I \in M\}$. Also, we have a natural p -PD morphism $\delta : \mathcal{D}_{(p)}(1) \rightarrow \mathcal{D}_{(p)}(1) \otimes_{\mathcal{O}_X} \mathcal{D}_{(p)}(1)$ such that $\delta(1 + \eta_i) = (1 + \eta_i) \otimes (1 + \eta_i)$, which allows us to define composition on the ring $\hat{\mathcal{D}}_{(p),X/S} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}_{(p)}(1), \mathcal{O}_X)$ of p -HPD differential operators $\mathcal{O}_X \rightarrow \mathcal{O}_X$.

PROPOSITION 4.1. *The following data are equivalent:*

- i) An object E of $\text{MIC}_{PD}(X/S)$.
- ii) A ring homomorphism $\hat{\mathcal{D}}_{(p),X/S} \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}_{(p)}(1) \otimes E, E)$.
- iii) A p -HPD stratification $\varepsilon : \mathcal{D}_{(p)}(1) \otimes E \rightarrow E \otimes \mathcal{D}_{(p)}(1)$ satisfying the natural cocycle condition.
- iv) A crystal E of $\mathcal{O}_{X/S}$ -modules on the site $\text{Crys}_{(p)}(X/S)$.

Proof. The proof follows as in the case of regular crystals once we establish an isomorphism $D_{X/S}^\gamma \simeq \hat{\mathcal{D}}_{(p),X/S}$. To define this isomorphism, first note that we have a natural map $D(1) \rightarrow D_{(p)}(1)$ since $D(1)$ is a member of $\text{Crys}_{(p)}(X/S)$; this induces a ring homomorphism $\mathcal{D}_{X/S} \rightarrow \hat{\mathcal{D}}_{(p),X/S}$ by pushforward. Also, if I is the p -PD ideal of $\mathcal{D}_{0,(p)}(1)$, then we have a natural isomorphism $S \cdot F_{X/S}^* \Omega_{X'/S}^1 \xrightarrow{\sim} \mathcal{D}_{(p)}(1)/I\mathcal{D}_{(p)}(1)$ defined by $1 \otimes \pi^* \omega \mapsto \omega^{[p]}$. This gives a natural isomorphism $\hat{\Gamma} \cdot F_{X/S}^* \mathcal{T}_{X'/S} \simeq (I\mathcal{D}_{(p)}(1))^\perp$. Furthermore, since $\delta(\omega^{[p]}) \equiv \omega^{[p]} \otimes 1 + 1 \otimes \omega^{[p]} \pmod{I\mathcal{D}_{(p)}(1)}$ for $\omega \in \Omega_{X/S}^1$, δ becomes the comultiplication on $S \cdot F_{X/S}^* \Omega_{X'/S}^1$, and since $1 \otimes f - f \otimes 1 \in I$ for $f \in \mathcal{O}_X$, composition is dual to this comultiplication map. This implies that the map $\hat{\Gamma} \cdot F_{X/S}^* \mathcal{T}_{X'/S} \simeq (I\mathcal{D}_{(p)}(1))^\perp$ is in fact a ring homomorphism. It is easy to see the two maps agree on $S \cdot F_{X/S}^* \mathcal{T}_{X'/S}$, and a local calculation below will show that the image of $\mathcal{D}_{X/S}$ commutes with $(I\mathcal{D}_{(p)}(1))^\perp$, so we get a map $D_{X/S}^\gamma \rightarrow \hat{\mathcal{D}}_{(p),X/S}$.

To prove this gives an isomorphism, it suffices to work locally, so assume we have a system of logarithmic coordinates m_1, \dots, m_r . Define $\zeta^{(I+pJ)} := \zeta^I (\zeta^{[p]})^J / I!$ for $I \in M$ and $J \in \mathbb{N}^r$, and let $\{D_{(N)}\}$ be the dual basis of $\hat{\mathcal{D}}_{(p),X/S}$ to $\{\zeta^{(N)}\}$. Now $\{D_I \otimes (D')^{[J]} : I \in M, J \in \mathbb{N}^r\}$ forms a completion basis for $D_{X/S}^\gamma$; we claim that our map sends $D_I \otimes (D')^{[J]}$ to $D_{(I+pJ)}$, which will complete

the proof. It is easy to check that the map $\mathcal{D}_{X/S} \rightarrow \hat{\mathcal{D}}_{(p),X/S}$ sends D_I to $D_{\langle I \rangle}$ for $I \in M$, and the map $\hat{\Gamma}.F_{X/S}^* \mathcal{T}_{X'/S} \rightarrow \hat{\mathcal{D}}_{(p),X/S}$ sends $(D')^{[J]}$ to $D_{\langle pJ \rangle}$; thus, it will suffice to show $D_{\langle I \rangle} \circ D_{\langle pJ \rangle} = D_{\langle pJ \rangle} \circ D_{\langle I \rangle} = D_{\langle I+pJ \rangle}$. However, for $A \in M$, we have $\delta(\eta^{\langle A \rangle}) = (\eta \otimes 1 + 1 \otimes \eta + \eta \otimes \eta)^A/A!$. If we reduce modulo $1 \otimes ID_{(p)}(1)$, the only nonzero term in the expansion of this expression is $(\eta \otimes 1)^A/A! = \eta^{\langle A \rangle} \otimes 1$. Thus, for $A \in M$ and $B \in \mathbb{N}^r$,

$$\begin{aligned} \delta(\eta^{\langle A+pB \rangle}) &\equiv (\eta^{\langle A \rangle} \otimes 1)(\eta^{[p]} \otimes 1 + 1 \otimes \eta^{[p]})^B \\ &\equiv \sum_{D \leq B} \binom{B}{D} \eta^{\langle A+p(B-D) \rangle} \otimes \eta^{\langle pD \rangle} \pmod{1 \otimes ID_{(p)}(1)}. \end{aligned}$$

Applying $\text{id} \otimes D_{\langle pJ \rangle}$ gives $\eta^{\langle A+p(B-J) \rangle}$, and applying $D_{\langle I \rangle}$ to this gives $\delta_{AI} \delta_{BJ} = \delta_{A+pB, I+pJ}$; this shows that $D_{\langle I \rangle} \circ D_{\langle pJ \rangle} = D_{\langle I+pJ \rangle}$. Similarly, reducing $\delta(\eta^{\langle A \rangle})$ modulo $ID_{(p)}(1) \otimes 1$ gives $1 \otimes \eta^{\langle A \rangle}$, so reducing $\delta(\eta^{\langle A+pB \rangle})$ gives $\sum_{D \leq B} \binom{B}{D} \eta^{\langle p(B-D) \rangle} \otimes \eta^{\langle A+pD \rangle}$. Hence $D_{\langle pJ \rangle} \circ D_{\langle I \rangle}(\eta^{\langle A+pB \rangle}) = \delta_{AI} \delta_{BJ}$, and $D_{\langle pJ \rangle} \circ D_{\langle I \rangle} = D_{\langle I+pJ \rangle}$. \square

Note that in particular the constructions from the previous chapter easily extend to give crystals $\mathcal{E}_{X/S}$ and $\mathcal{K}_{X/S}$ on $\text{Crys}_{(p)}(X/S)$, so they have canonical structures of $D_{X/S}^\gamma$ -module. In particular, for $(U, \tilde{T}, \tilde{V}) \in \text{Crys}_{(p)}(X/\tilde{S})$, $\mathcal{E}_{X/S, \tilde{V}}$ is the conormal sheaf of $V \xrightarrow{\Gamma} V_S X' \rightarrow \tilde{V}_{\tilde{S}} \tilde{X}'$, where Γ is the graph of $f_{V/S}$, the composition of the map $V \rightarrow T$ with $f_{T/S}$. Similarly, $MIC_{PD}^A(X/S)$ is equivalent to the category of p -crystals of $\mathcal{A}_X^{\text{gp}}$ -modules.

4.2 The Global Cartier Transform

Given a lifting $\tilde{X}' \rightarrow \tilde{S}$ of X' over S , we locally have isomorphisms $\mathcal{K}_{X/S} \simeq S^* F_{X/S}^* \Omega_{X'/S}^1$, and thus $\check{\mathcal{K}}_{X/S} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{K}_{X/S}, \mathcal{O}_X) \simeq \hat{\Gamma}.F_{X/S}^* \mathcal{T}_{X'/S} \simeq F_{X/S}^* \mathcal{O}_{\mathcal{G}}$. Now the total connection on $\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \check{\mathcal{K}}_{X/S}$ extends naturally to a $\tilde{\mathcal{D}}_{X/S}^\gamma$ -module structure, and $\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \check{\mathcal{K}}_{X/S}$ is a locally free $\mathcal{O}_{\mathcal{G}}^A$ -module of rank 1 (with basis $1 \otimes \xi_{\tilde{F}}$ for \tilde{F} a lifting of $F_{X/S}$). However, $\mathcal{O}_{\mathcal{G}}^A \simeq \mathcal{O}_{\mathcal{G}}^B \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}}$ is locally free of rank p^d over $\mathcal{O}_{\mathcal{G}}^B$. Hence $\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \check{\mathcal{K}}_{X/S}$ is a splitting module for $\tilde{\mathcal{D}}_{X/S}$ over $\mathcal{O}_{\mathcal{G}}^B$, immediately giving the following equivalence.

THEOREM 4.2. *Let $\check{\mathcal{K}}_{X/S}^A := \mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \check{\mathcal{K}}_{X/S}$, and define functors*

$$\begin{aligned} C_{X/S} : MIC_{PD}^A(X/S) &\rightarrow HIG_{PD}^B(X'/S), E \mapsto \iota_* \mathcal{H}om_{\tilde{\mathcal{D}}_{X/S}^\gamma}(\check{\mathcal{K}}_{X/S}^A, E), \\ C_{X/S}^{-1} : HIG_{PD}^B(X'/S) &\rightarrow MIC_{PD}^A(X/S), E' \mapsto \check{\mathcal{K}}_{X/S}^A \otimes_{\mathcal{O}_{\mathcal{G}}^B} \iota_* E'. \end{aligned}$$

(Here $\iota : \mathcal{O}_{\mathcal{G}} \rightarrow \mathcal{O}_{\mathcal{G}}$ is the map corresponding to the inverse map $\mathcal{G} \rightarrow \mathcal{G}$, so that $\iota_* E'$ is E' with a sign change of the Higgs field.) Then $C_{X/S}$ and $C_{X/S}^{-1}$ are quasi-inverse equivalences of categories.

Remark 4.3. We will see later that the sign changes in the above definitions are necessary in order for the isomorphism of de Rham and Higgs complexes to be compatible with the standard Cartier isomorphism.

In the case that $\mathcal{J} = \overline{\mathcal{M}}_X^{\text{gp}}$ with the standard action, we have that the category of \mathcal{J} -indexed $\mathcal{A}_X^{\text{gp}}$ -modules is equivalent to the category of \mathcal{O}_X -modules; indeed, it is easy to see that for any $\mathcal{A}_X^{\text{gp}}$ -module E , the natural map $E_0 \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}} \rightarrow E$ is an isomorphism. Similarly, a connection on E is admissible if and only if this map is horizontal. We thus get:

THEOREM 4.4. *Given a lifting X'/S of X' over S , we have an equivalence of categories*

$$C_{X/S} : MIC_{PD}(X/S) \rightarrow HIG_{PD}^B(X'/S).$$

However, $\check{\mathcal{K}}_{\mathcal{X}/S}^A$ has the unfortunate property that it is not locally nilpotent. We thus reformulate the functors described above in terms of $\mathcal{K}_{\mathcal{X}/S}^A := \mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} \mathcal{K}_{\mathcal{X}/S}$ instead. For E an object of $MIC_{PD}^A(X/S)$, let $E^\theta := \mathcal{H}om_{\mathcal{O}_G^A}(\mathcal{A}_X^{\text{gp}}, E)$ be the subsheaf of E annihilated by $\Gamma_+ \mathcal{T}_{X'/S}$, and let $E^{\nabla, \gamma} := \mathcal{H}om_{\tilde{\mathcal{D}}_{X/S}^\gamma}(\mathcal{A}_X^{\text{gp}}, E)$ be the subsheaf of E^θ annihilated by ∇ .

THEOREM 4.5. i) For E an object of $MIC_{PD}^A(X/S)$, there is a natural isomorphism

$$C_{\mathcal{X}/S}(E) \simeq (\mathcal{K}_{\mathcal{X}/S}^A \otimes_{\mathcal{A}_X^{\text{gp}}} E)^{\nabla, \gamma},$$

where \mathcal{O}_G^B acts on the right hand side via the action on $\mathcal{K}_{\mathcal{X}/S}^A$.

ii) For E' an object of $HIG_{PD}^B(X'/S)$, there is a natural isomorphism

$$C_{\mathcal{X}/S}^{-1}(E') \simeq (\mathcal{K}_{\mathcal{X}/S}^A \otimes_{\mathcal{B}_{X/S}} E')^\theta,$$

where $\tilde{\mathcal{D}}_{X/S}^\gamma$ acts on the right hand side via the action on $\mathcal{K}_{\mathcal{X}/S}^A$.

Proof. First, since E is locally nilpotent, and $\check{\mathcal{K}}_{\mathcal{X}/S}^A$ is locally free of rank 1 as an \mathcal{O}_G^A -module, we have an isomorphism

$$\mathcal{H}om_{\mathcal{O}_G^A}(\check{\mathcal{K}}_{\mathcal{X}/S}^A, E) \simeq (\mathcal{K}_{\mathcal{X}/S}^A \otimes_{\mathcal{A}_X^{\text{gp}}} E)^\theta.$$

Under this isomorphism, the submodule $C_{\mathcal{X}/S}(E) = \mathcal{H}om_{\tilde{\mathcal{D}}_{X/S}^\gamma}(\check{\mathcal{K}}_{\mathcal{X}/S}^A, E)$ corresponds to $(\mathcal{K}_{\mathcal{X}/S}^A \otimes_{\mathcal{A}_X^{\text{gp}}} E)^{\nabla, \gamma}$ with \mathcal{O}_G^B acting on E . Changing the sign is the same as letting \mathcal{O}_G^B act on $\mathcal{K}_{\mathcal{X}/S}^A$ instead, proving (i).

To prove (ii), we begin with a lemma.

LEMMA 4.6. *There is a canonical isomorphism*

$$\check{\mathcal{K}}_{\mathcal{X}/S} \otimes_{F^* \mathcal{O}_G} \iota_* \check{\mathcal{K}}_{\mathcal{X}/S} \simeq F_{X/S}^* \mathcal{O}_G.$$

Proof. Suppose we have a lifting \tilde{F} of $F_{X/S}$. Then $\xi_{\tilde{F}} \otimes \xi_{\tilde{F}}$ forms a basis for $\check{\mathcal{K}}_{\mathcal{X}/S} \otimes_{F^* \mathcal{O}_G} \iota_* \check{\mathcal{K}}_{\mathcal{X}/S}$ as an $F_{X/S}^* \mathcal{O}_G$ -module. We claim that, in fact, this basis element is independent of the choice of \tilde{F} , so the local bases glue to a global basis of $\check{\mathcal{K}}_{\mathcal{X}/S} \otimes_{F^* \mathcal{O}_G} \iota_* \check{\mathcal{K}}_{\mathcal{X}/S}$. Thus, suppose we have two liftings \tilde{F}_1 and \tilde{F}_2 , and let $h : F_{X/S}^* \Omega_{X'/S}^1 \rightarrow \mathcal{O}_X$ express the difference between them. Also, suppose we have a system of logarithmic coordinates $m'_1, \dots, m'_r \in \mathcal{M}_{X'}^{\text{gp}}$, and let $D^{[N]}$ ($N \in \mathbb{N}^r$) be the corresponding basis of $F_{X/S}^* \mathcal{O}_G = F_{X/S}^* \hat{\Gamma} \cdot \mathcal{T}_{X'/S}$. Then for $N \in \mathbb{N}^r$,

$$\prod_{i=1}^r \sigma_{\tilde{F}_2}(d \log(m'_i))^{N_i} = \prod_{i=1}^r (\sigma_{\tilde{F}_1}(d \log(m'_i)) + h(d \log(m'_i)))^{N_i}.$$

Now $\xi_{\tilde{F}_1}$ maps this element of $\mathcal{K}_{\mathcal{X}/S}$ to $h^N := \prod_{i=1}^r h(d \log(m'_i))^{N_i}$, while $\theta_{D^{[N]}} \xi_{\tilde{F}_2}$ maps it to $(-1)^{|I|} \delta_{IN}$. In other words,

$$\xi_{\tilde{F}_1} = \sum_{N \in \mathbb{N}^r} (-1)^{|N|} h^N \theta_{D^{[N]}} \xi_{\tilde{F}_2}.$$

Thus, defining $g := \sum_N (-1)^{|N|} h^N D^{[N]} \in F_{X/S}^* \mathcal{O}_G$, we see that $\xi_{\tilde{F}_1} = \theta_g \xi_{\tilde{F}_2}$; also, plugging in $-h$ in place of h , we get $\xi_{\tilde{F}_2} = \theta_{\iota(g)} \xi_{\tilde{F}_1}$. Therefore,

$$\xi_{\tilde{F}_1} \otimes \xi_{\tilde{F}_1} = \theta_g \xi_{\tilde{F}_2} \otimes \xi_{\tilde{F}_1} = \xi_{\tilde{F}_2} \otimes \theta_{\iota(g)} \xi_{\tilde{F}_1} = \xi_{\tilde{F}_2} \otimes \xi_{\tilde{F}_2}.$$

□

Now for E' an object of $HIG_{PD}^B(X'/S)$, let $E_0 := E' \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}}$. Using the above result, we have $\check{\mathcal{K}}_{X/S}^A \otimes_{\mathcal{O}_G^A} \iota_* \check{\mathcal{K}}_{X/S}^A \simeq \mathcal{O}_G^A$. Hence

$$\begin{aligned} \check{\mathcal{K}}_{X/S}^A \otimes_{\mathcal{O}_G^B} \iota_* E' &\simeq \check{\mathcal{K}}_{X/S}^A \otimes_{\mathcal{O}_G^A} \iota_* E_0 \simeq \mathcal{H}om_{\mathcal{O}_G^A}(\iota_* \check{\mathcal{K}}_{X/S}^A, \iota_* E_0) \simeq \mathcal{H}om_{\mathcal{O}_G^A}(\check{\mathcal{K}}_{X/S}^A, E_0) \\ &\simeq (\mathcal{K}_{X/S}^A \otimes_{\mathcal{A}_X^{\text{gp}}} E_0)^\theta \simeq (\mathcal{K}_{X/S}^A \otimes_{\mathcal{B}_{X/S}} E')^\theta. \end{aligned}$$

□

As an application, we can use this to calculate $C_{X/S}(\mathcal{K}_{X/S}^A)$. First, we claim there is a canonical isomorphism

$$(\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_{X/S})^\theta \simeq S^* F_{X/S}^* \Omega_{X'/S}^1 \simeq F_{X/S}^* (\iota_* \check{\mathcal{O}}_G).$$

To construct this isomorphism, we begin with the cosubtraction map $S^* F_{X/S}^* \Omega_{X'/S}^1 \rightarrow \mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_{X/S}$. Then locally on X , if we have a lifting \tilde{F} of $F_{X/S}$, the restriction to $F_{X/S}^* \Omega_{X'/S}^1 \rightarrow \mathcal{E}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}_{X/S}$ is equal to $1 \otimes \sigma_{\tilde{F}} - \sigma_{\tilde{F}} \otimes 1$, which factors through $(\mathcal{E}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}_{X/S})^\theta$. Since the total \mathcal{G} -Higgs field on $\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_{X/S}$ satisfies the Leibniz rule, this implies that the cosubtraction map factors through $(\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_{X/S})^\theta$.

We may check the map $S^* F_{X/S}^* \Omega_{X'/S}^1 \rightarrow (\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_{X/S})^\theta$ is an isomorphism locally, so we may assume there is a lifting \tilde{F} of $F_{X/S}$. Then $(\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_{X/S})^\theta \simeq \mathcal{H}om_{F^* \mathcal{O}_G}(\check{\mathcal{K}}_{X/S}, \mathcal{K}_{X/S})$, and $\xi_{\tilde{F}}$ forms a basis for $\check{\mathcal{K}}_{X/S}$ as an $F^* \mathcal{O}_G$ -module. Thus, $\xi_{\tilde{F}} \otimes \text{id}$ gives an isomorphism $(\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_{X/S})^\theta \simeq \mathcal{K}_{X/S}$, and the composition with the above map is just the isomorphism $S^* F_{X/S}^* \Omega_{X'/S}^1 \xrightarrow{\sim} \mathcal{K}_{X/S}$ induced by \tilde{F} . In fact, this shows that the canonical structure of \mathcal{G} -Higgs field on $F_{X/S}^* (\iota_* \check{\mathcal{O}}_G)$ corresponds to the action of \mathcal{O}_G on the second $\mathcal{K}_{X/S}$. Note also that since $\nabla_{\text{tot}}(1 \otimes \sigma_{\tilde{F}}(1 \otimes \omega) - \sigma_{\tilde{F}}(1 \otimes \omega) \otimes 1) = -(1 \otimes 1) \otimes \zeta_{\tilde{F}}(\omega) + (1 \otimes 1) \otimes \zeta_{\tilde{F}}(\omega) = 0$, the total connection on $(\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_{X/S})^\theta$ corresponds to the Frobenius descent connection on $F_{X/S}^* (\iota_* \check{\mathcal{O}}_G)$. From this it follows that there is a canonical isomorphism $C_{X/S}(\mathcal{K}_{X/S}^A) \simeq (\mathcal{K}_{X/S}^A \otimes_{\mathcal{A}_X^{\text{gp}}} \mathcal{K}_{X/S}^A)^\theta \simeq \iota_* \check{\mathcal{O}}_G \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S}$.

4.3 Nilpotent Residue

The goal of this section is to apply the preceding theory to get information about objects of $MIC_{PD}(X/S)$ and of $HIG_{PD}(X'/S)$. To do this, note that for E an object of $MIC_{PD}(X/S)$, $E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}$ with the total connection is an object of $MIC_{PD}^A(X/S)$; similarly, for E' an object of $HIG_{PD}(X'/S)$, $E' \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S}$ is an object of $HIG_{PD}^B(X'/S)$. Our first task is to calculate what the Cartier transform does to these objects.

PROPOSITION 4.7. i) Let E' be an object of $HIG_{PD}(X'/S)$. Then there is a natural isomorphism

$$C_{X/S}^{-1}(E' \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S}) \simeq (\check{\mathcal{K}}_{X/S} \otimes_{\mathcal{O}_G} \iota_* E') \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}.$$

ii) Let E be an object of $MIC_{PD}(X/S)$. Then there is a natural map

$$\iota_* \mathcal{H}om_{\mathcal{D}_{X/S}^\gamma}(\check{\mathcal{K}}_{X/S}, E) \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S} \rightarrow C_{X/S}(E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}).$$

Furthermore, denoting $E_0 := \mathcal{H}om_{F^* \mathcal{O}_G}(\check{\mathcal{K}}_{X/S}, E)$ with the internal Hom connection, this map is injective, resp. surjective, if and only if the natural map $E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \rightarrow E_0$ is.

Example 4.8. To illustrate the second part, let X be \mathbb{A}_k^1 minus one point with k a field of characteristic p ; that is, $X := \text{Spec}(\mathbb{N} \rightarrow k[t])$, where the map $\mathbb{N} \rightarrow k[t]$ sends 1 to t , and $S := \text{Spec } k$. Consider the \mathcal{O}_X -module $\mathcal{O}_X \cdot e$ with the connection ∇ such that $\nabla(e) = e \otimes d \log(t)$. Then since ∇ has zero p -curvature, any $F_{X/S}^* \mathcal{O}_G$ -linear map $\check{\mathcal{K}}_{X/S} \rightarrow E$ factors uniquely through the map $\check{\mathcal{K}}_{X/S} \rightarrow \mathcal{O}_X$ induced by the inclusion $\mathcal{O}_X \rightarrow \mathcal{K}_{X/S}$. We thus get an isomorphism $E_0 \simeq E$, and we

will see below that this isomorphism is horizontal. Thus $\mathcal{H}om_{\mathcal{D}_{X/S}^\gamma}(\check{\mathcal{K}}_{X/S}, E) \simeq E^\nabla = \mathcal{O}_{X'} \cdot t^{p-1}e$. Similarly, $C_{X/S}(E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}) \simeq (E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}})^\nabla = \mathcal{B}_{X/S} \cdot (e \otimes e_{-1})$. Now the map described in the proof of (ii) below sends $t^{p-1}e$ to $t^{p-1}e \otimes 1 = (t^{p-1}e_1) \cdot (e \otimes e_{-1})$, where $t^{p-1}e_1 \in \mathcal{B}_{X/S}$. This map is clearly not surjective, and neither is the map $E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \rightarrow E_0$.

Proof. For (i), we have

$$\begin{aligned} C_{X/S}^{-1}(E' \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S}) &= \check{\mathcal{K}}_{X/S}^A \otimes_{\mathcal{O}_G^B} (\iota_* E' \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S}) \simeq \check{\mathcal{K}}_{X/S}^A \otimes_{\mathcal{O}_G} \iota_* E' \\ &\simeq (\check{\mathcal{K}}_{X/S} \otimes_{\mathcal{O}_G} \iota_* E') \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}, \end{aligned}$$

as required.

For (ii), we have $C_{X/S}(E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}) \simeq \iota_* \mathcal{H}om_{\tilde{\mathcal{D}}_{X/S}^\gamma}(\check{\mathcal{K}}_{X/S}^A, E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}})$. However, since

$$\mathcal{H}om_{\mathcal{O}_G^A}(\check{\mathcal{K}}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}, E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}) \simeq \mathcal{H}om_{F^* \mathcal{O}_G}(\check{\mathcal{K}}_{X/S}, E) \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}} = E_0 \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}},$$

applying ι_* to the natural map $E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S} \rightarrow (E_0 \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}})^\nabla$ gives the desired map.

We now have a commutative diagram

$$\begin{array}{ccc} (E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S}) \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}} & \longrightarrow & (E_0 \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}})^\nabla \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}} \\ \downarrow \sim & & \downarrow \sim \\ (E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}} & \longrightarrow & E_0 \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}. \end{array}$$

Here the map on the right is an isomorphism since E_0 has zero p -curvature by definition. Now the top row is injective, resp. surjective, if and only if $E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S} \rightarrow (E_0 \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}})^\nabla$ is, since $\mathcal{A}_X^{\text{gp}}$ is locally free over $\mathcal{B}_{X/S}$. Similarly, the bottom row is injective, resp. surjective, if and only if $E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \rightarrow E_0$ is, since $(\mathcal{A}_X^{\text{gp}})_s$ is invertible for each section $s \in \overline{\mathcal{M}}_X^{\text{gp}}$. \square

We now need to study the map $E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \rightarrow E_0$; however, the p -curvature of ∇ on E_0 is zero since each homomorphism in E_0 commutes with $\mathcal{T}_{X'/S} \subseteq \mathcal{O}_G$. Recall that for a log scheme X over S , we define the *residue sheaf* $\mathcal{R}_{X/S} := \Omega_{X/X^*}^1$, where X^* denotes the log scheme with the same underlying scheme as X but with log structure $f^* \mathcal{M}_S$; we then have an exact sequence

$$\Omega_{X^*/S}^1 \rightarrow \Omega_{X/S}^1 \rightarrow \mathcal{R}_{X/S} \rightarrow 0.$$

Also, for a connection $\nabla : E \rightarrow E \otimes_{\mathcal{O}_X} \Omega_{X/S}^1$ we define the residue to be the induced map $\rho : E \rightarrow E \otimes_{\mathcal{O}_X} \mathcal{R}_{X/S}$; for each derivation $D \in \mathcal{T}_{X/S}$ this induces an endomorphism ρ_D on $E/D(\mathcal{O}_X)E$, where $D(\mathcal{O}_X)$ is the ideal of \mathcal{O}_X generated by the image of D . Now according to [Lor00], the map $E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \rightarrow E_0$ is surjective if the residue of the connection on E_0 vanishes, and it is an isomorphism if in addition $\text{Tor}_1(E_0, \mathcal{R}_{X/S}) = 0$. However, since $\check{\mathcal{K}}_{X/S}$ has basis $\xi_{\tilde{F}}$ as an $F_{X/S}^* \mathcal{O}_G$ -module, E_0 is locally isomorphic to E ; we now calculate the connection on E_0 in terms of this isomorphism.

PROPOSITION 4.9. *Suppose we are given a lifting \tilde{F} of $F_{X/S}$, and let ∇' be the connection on E which makes the following diagram commute:*

$$\begin{array}{ccc} \mathcal{H}om_{F^* \mathcal{O}_G}(\check{\mathcal{K}}_{X/S}, E) & \xrightarrow[\sim]{\cdot(\xi_{\tilde{F}})} & E \\ \downarrow \nabla & & \downarrow \nabla' \\ \mathcal{H}om_{F^* \mathcal{O}_G}(\check{\mathcal{K}}_{X/S}, E) \otimes \Omega_{X/S}^1 & \xrightarrow[\sim]{\cdot(\xi_{\tilde{F}}) \otimes \text{id}} & E \otimes \Omega_{X/S}^1. \end{array}$$

i) We have

$$\nabla' = \nabla + (\text{id} \otimes \zeta_{\tilde{F}}) \circ \psi,$$

where the second term is the composition $E \xrightarrow{\psi} E \otimes_{\mathcal{O}_X} F_{X/S}^* \Omega_{X'/S}^1 \xrightarrow{\text{id} \otimes \zeta_{\tilde{F}}} E \otimes_{\mathcal{O}_X} \Omega_{X/S}^1$.

- ii) Also suppose we have a logarithmic system of coordinates $m_1, \dots, m_r \in \mathcal{M}_X^{\text{gp}}$ for X , and let D_1, \dots, D_r be the corresponding basis of $\mathcal{T}_{X/S}$. Then $\rho'_{D_i} = \rho_{D_i}^p$ for $i = 1, \dots, r$.

Proof. To prove (i), we first calculate the connection on $\tilde{\mathcal{K}}_{X/S}$: we see that $(\nabla \xi_{\tilde{F}})(1) = 0$; $(\nabla \xi_{\tilde{F}})(\sigma_{\tilde{F}}(\pi^* d \log(m_i))) = -1 \otimes \zeta_{\tilde{F}}(\pi^* d \log(m_i))$; and $\nabla \xi_{\tilde{F}}$ is zero on $S^{\geq 2} \sigma_{\tilde{F}}(F_{X/S}^* \Omega_{X'/S}^1)$. Similarly, $(\psi \xi_{\tilde{F}})(1) = 0$; $(\psi \xi_{\tilde{F}})(\sigma_{\tilde{F}}(\pi^* d \log(m_j))) = 1 \otimes d \log(m_j)$; and $\psi \xi_{\tilde{F}}$ is zero on $S^{\geq 2} \sigma_{\tilde{F}}(F_{X/S}^* \Omega_{X'/S}^1)$. Therefore, $\nabla \xi_{\tilde{F}} = -(\text{id} \otimes \zeta_{\tilde{F}})(\psi \xi_{\tilde{F}})$. Now consider an element $\phi \in \mathcal{H}om_{F^* \mathcal{O}_G}(\tilde{\mathcal{K}}_{X/S}, E)$, and let $e := \phi(\xi_{\tilde{F}})$. Then

$$\begin{aligned} \nabla' e &= (\nabla \phi)(\xi_{\tilde{F}}) = \nabla(\phi(\xi_{\tilde{F}})) - (\phi \otimes \text{id})(\nabla \xi_{\tilde{F}}) \\ &= \nabla e - (\phi \otimes \text{id}) \left(- \sum_{i=1}^r \psi_{D'_i} \xi_{\tilde{F}} \otimes \zeta_{\tilde{F}}(\pi^* d \log(m_i)) \right) \\ &= \nabla e + \sum_{i=1}^r \psi_{D'_i} e \otimes \zeta_{\tilde{F}}(\pi^* d \log(m_i)) \\ &= \nabla e + (\text{id} \otimes \zeta_{\tilde{F}})(\psi e), \end{aligned}$$

as required.

To prove (ii), note that $\zeta_{\tilde{F}}(\pi^* d \log(m_j)) = d \log(m_j) + db_j$ for some $b_j \in \mathcal{O}_X$. Thus, $\langle \zeta_{\tilde{F}}(\pi^* d \log(m_j)), D_i \rangle = \delta_{ij} + D_i b_j$, so from (i) it follows that $\rho'_{D_i} = \rho_{D_i} + \bar{\psi}_{D'_i}$, where $\bar{\psi}$ is the residue of the p -curvature. However, $\psi_{D'_i} = \nabla_{D'_i}^p - \nabla_{D_i}$, so $\bar{\psi}_{D'_i} = \rho_{D_i}^p - \rho_{D_i}$, completing the proof. \square

Remark 4.10. If ∇ is locally nilpotent, the following local characterization of $E_0 \simeq (\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} E)^\theta$ is useful for calculations: suppose we have a lifting \tilde{F} of $F_{X/S}$, which we use to identify $\mathcal{K}_{X/S}$ with $S^* F_{X/S}^* \Omega_{X'/S}^1$, and a logarithmic system of coordinates $m_1, \dots, m_r \in \mathcal{M}_X^{\text{gp}}$. Then $(\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} E)^\theta$ is the set of elements of $\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} E$ of the form

$$\sum_{N \in \mathbb{N}^r} (-1)^{|N|} (\pi^* d \log m)^N \otimes \theta_{(D')^{[N]}}(e)$$

for some $e \in E$, where D'_1, \dots, D'_r is the basis of $\mathcal{T}_{X'/S}$ dual to the basis $\pi^* d \log m_1, \dots, \pi^* d \log m_r$ of $\Omega_{X'/S}^1$.

COROLLARY 4.11. *Let $MIC_{PD}^0(X/S)$ be the full subcategory of $MIC_{PD}(X/S)$ consisting of objects (E, ∇, θ) such that the residue ρ of ∇ satisfies $\rho_D^p = 0$ for every $D \in \mathcal{T}_{X/S}$. Define functors $C_{X/S} : MIC_{PD}^0(X/S) \rightarrow HIG_{PD}(X'/S)$ and $C_{X/S}^{-1} : HIG_{PD}(X'/S) \rightarrow MIC_{PD}^0(X/S)$ by*

$$\begin{aligned} C_{X/S}(E) &:= \iota_* \mathcal{H}om_{\mathcal{D}_{X/S}^\gamma}(\tilde{\mathcal{K}}_{X/S}, E), \\ C_{X/S}^{-1}(E') &:= \tilde{\mathcal{K}}_{X/S} \otimes_{\mathcal{O}_G} \iota_* E' \end{aligned}$$

- i) The functor $C_{X/S}$ has left adjoint $C_{X/S}^{-1}$.
- ii) The unit $\eta : \text{id} \rightarrow C_{X/S} \circ C_{X/S}^{-1}$ of this adjunction is an isomorphism, and the counit $\epsilon : C_{X/S}^{-1} \circ C_{X/S} \rightarrow \text{id}$ is an epimorphism on every object of $MIC_{PD}^0(X/S)$. (Note that this implies that $C_{X/S}$ is faithful, and $C_{X/S}^{-1}$ is fully faithful.)
- iii) If $\text{Tor}_1(E, \mathcal{R}_{X/S}) = 0$, then ϵ_E is an isomorphism. (Hence E is in the essential image of $C_{X/S}^{-1}$.)

Proof. Note that $\tilde{\mathcal{K}}_{X/S}$ is itself an object of $MIC_{PD}^0(X/S)$, so $C_{X/S}^{-1}$ does indeed have image contained in $MIC_{PD}^0(X/S)$. Now the adjunction in (i) is simply the standard adjunction between $\tilde{\mathcal{K}}_{X/S} \otimes \cdot$ and $\mathcal{H}om_{\mathcal{D}_{X/S}^\gamma}(\tilde{\mathcal{K}}_{X/S}, \cdot)$ discussed in (2.3).

For $E \in MIC_{PD}^0(X/S)$, letting $E_0 := \mathcal{H}om_{F^*\mathcal{O}_G}(\check{\mathcal{K}}_{X/S}, E)$, we see that the residue of ∇_{tot} on E_0 is zero. Thus, we have a natural surjection $C_{X/S}(E) \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S} \rightarrow C_{X/S}(E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}})$, which is an isomorphism if $\text{Tor}_1(E_0, \mathcal{R}_{X/S}) = 0$. However, since E_0 is locally isomorphic to E , this is equivalent to the condition that $\text{Tor}_1(E, \mathcal{R}_{X/S}) = 0$. Now applying $C_{X/S}^{-1}$ to this map gives a surjection $C_{X/S}^{-1}(C_{X/S}(E)) \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}} \rightarrow E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}$; however, it is straightforward to check that taking the degree zero part gives the counit ϵ .

Now for $E' \in HIG_{PD}(X'/S)$, let $E := C_{X/S}^{-1}(E') = \check{\mathcal{K}}_{X/S} \otimes_{\mathcal{O}_G} E'$. Then

$$E_0 := \mathcal{H}om_{F^*\mathcal{O}_G}(\check{\mathcal{K}}_{X/S}, E) \simeq \mathcal{H}om_{F^*\mathcal{O}_G}(\check{\mathcal{K}}_{X/S}, \check{\mathcal{K}}_{X/S} \otimes_{F^*\mathcal{O}_G} (\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} E')) \simeq \mathcal{O}_X \otimes_{\mathcal{O}_{X'}} E'$$

since $\check{\mathcal{K}}_{X/S}$ is invertible as an $F_{X/S}^*\mathcal{O}_G$ -module. Thus, $E_0^\nabla \simeq E'$, so $E_0 \simeq E_0^\nabla \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X$, and we get an isomorphism $C_{X/S}(E) \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S} \simeq C_{X/S}(E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}})$. On the other hand, $C_{X/S}^{-1}(E' \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S}) \simeq E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}}$, so $E' \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S} \simeq C_{X/S}(E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}})$. Again, we see that taking the degree zero part of the isomorphism $E' \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S} \simeq C_{X/S}(C_{X/S}^{-1}(E')) \otimes_{\mathcal{O}_{X'}} \mathcal{B}_{X/S}$ gives the unit η . \square

Similarly to before, we get isomorphisms $C_{X/S}(E) \simeq (\check{\mathcal{K}}_{X/S} \otimes_{\mathcal{O}_X} E)^\nabla, \gamma$ for $E \in MIC_{PD}^0(X/S)$, and $C_{X/S}^{-1}(E') \simeq (\check{\mathcal{K}}_{X/S} \otimes_{\mathcal{O}_{X'}} E')^\theta$ for $E' \in HIG_{PD}(X'/S)$.

4.4 The Local Cartier Transform

From the above calculations, we see that for $E \in MIC_{PD}^0(X'/S)$, given a lifting \tilde{F} of $F_{X/S}$ and a logarithmic system of coordinates $m_1, \dots, m_r \in \mathcal{M}_X^{\text{gp}}$, we have isomorphisms

$$C_{X/S}(E) \simeq (\iota_* E)^\nabla = \{x \in E : \nabla x = -(\text{id} \otimes \zeta_{\tilde{F}})(\psi x)\},$$

where ∇' is as in (4.9), with the PD Higgs field inherited from $-\psi$ on E . Similarly, for $E' \in HIG_{PD}(X'/S)$, the isomorphism $\check{\mathcal{K}}_{X/S} \simeq F_{X/S}^*\mathcal{O}_G$ induced by \tilde{F} gives an isomorphism $C_{X/S}^{-1}(E') \simeq F_{X/S}^*E'$, with the connection given by

$$\nabla(e' \otimes f) = e' \otimes df + (\text{id} \otimes \zeta_{\tilde{F}})(\psi' e') \otimes f.$$

Similar formulas hold for the equivalence between $MIC_{PD}^A(X/S)$ and $HIG_{PD}^B(X'/S)$. This last formula is our motivation for the following definition.

DEFINITION 4.12. Let $\zeta : \Omega_{X'/S}^1 \rightarrow F_{X/S*}\Omega_{X/S}^1$ be a splitting of the Cartier operator. We then define a functor $\Psi : MIC^A(X/S) \rightarrow F\text{-}HIG^A(X/S)$ which maps (E, ∇) to the p -curvature of ∇ on E . We also define a functor $\Psi_\zeta^{-1} : HIG^B(X'/S) \rightarrow MIC^A(X/S)$ by $\Psi_\zeta^{-1}(E', \psi') := E' \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}}$, with connection

$$\nabla(e' \otimes f) = e' \otimes df + (\text{id} \otimes \zeta)(\psi' e') \otimes f.$$

Similarly, we define functors $\Psi : MIC(X/S) \rightarrow F\text{-}HIG(X/S)$ by $\Psi(E, \nabla) := (E, \psi(\nabla))$ and $\Psi_\zeta^{-1} : HIG(X'/S) \rightarrow MIC(X/S)$ by $\Psi_\zeta^{-1}(E', \theta') := (F_{X/S}^*E', \nabla)$, where ∇ is given by the same formula as above.

We now calculate the p -curvature of this connection. In order to express the answer, we use the following notation: ζ gives a map $F_{X/S}^*\Omega_{X'/S}^1 \rightarrow \Omega_{X/S}^1$, so the adjoint ζ^* gives a map $\mathcal{T}_{X/S} \rightarrow F_{X/S}^*\mathcal{T}_{X'/S}$.

LEMMA 4.13. Let $(E', \theta') \in HIG^B(X'/S)$ or $HIG(X'/S)$.

- i) The connection ∇ on $\Psi_\zeta^{-1}(E', \theta')$ is integrable.
- ii) If ψ is the p -curvature of ∇ , then $\psi_{\pi^*D} = (F_{X/S}^*\theta')_{(\zeta^*D)^p - \pi^*D}$ for $D \in \mathcal{T}_{X/S}$.

Proof. The integrability of ∇ follows from the fact that the image of ζ consists of closed forms. To prove the formula for ψ , since $\Psi_\zeta^{-1}(E') \simeq E' \otimes_{S \cdot \mathcal{T}_{X'/S}} \Psi_\zeta^{-1}(S \cdot \mathcal{T}_{X'/S})$, it suffices to verify the formula for $E' = S \cdot \mathcal{T}_{X'/S}$. The calculation uses the following two identities.

LEMMA 4.14. [Lor00, 1.4.1] Let $\omega \in Z\Omega_{X/S}^1$ and $D \in \mathcal{T}_{X/S}$. Then

$$F_{X/S}^* \langle C_{X/S} \omega, \pi^* D \rangle = \langle \omega, D^{(p)} \rangle - D^{p-1} \langle \omega, D \rangle.$$

LEMMA 4.15. Let E' be an $\mathcal{O}_{X'}$ -module, and let $\alpha, \beta \in \mathcal{E}nd_{\mathcal{O}_{X'}}(E')$; now define β_n recursively by $\beta_0 = \beta$ and $\beta_{n+1} = [\alpha, \beta_n]$. Suppose that $\beta_0, \beta_1, \dots, \beta_{p-1}$ commute pairwise. Then

$$(\alpha + \beta)^p = \alpha^p + \beta_{p-1} + \beta^p.$$

Proof. Let P'_n be the set of pairs (S_0, π) , where S_0 is a subset of $\{1, 2, \dots, n\}$ and π is a partition of $\{1, 2, \dots, n\} - S_0$. We then prove by induction on n that

$$(\alpha + \beta)^n = \sum_{(S_0, \pi) \in P'_n} \left(\prod_{S \in \pi} \beta_{|S|-1} \right) \alpha^{|S_0|}.$$

For $n = 0$, the statement is trivial. Now for the inductive step, we use the identity $\alpha(\phi_1 \phi_2 \cdots \phi_n) = [\alpha, \phi_1] \phi_2 \cdots \phi_n + \phi_1 [\alpha, \phi_2] \cdots \phi_n + \cdots + \phi_1 \phi_2 \cdots [\alpha, \phi_n] + \phi_1 \phi_2 \cdots \phi_n \alpha$. Applying this to a term in the sum above, we see that changing $\beta_{|S|-1}$ to $[\alpha, \beta_{|S|-1}] = \beta_{|S|}$ corresponds to adding $n+1$ to S ; multiplying on the right by α corresponds to adding $n+1$ to S_0 ; and multiplying on the left by β corresponds to adding $\{n+1\}$ to π . Thus

$$(\alpha + \beta) \sum_{(S_0, \pi) \in P'_n} \left(\prod_{S \in \pi} \beta_{|S|-1} \right) \alpha^{|S_0|} = \sum_{(S_0, \pi) \in P'_{n+1}} \left(\prod_{S \in \pi} \beta_{|S|-1} \right) \alpha^{|S_0|},$$

completing the induction.

Now considering the action of the cycle $(1, 2, \dots, p)$ on P'_p , we see that in the expression for $(\alpha + \beta)^p$, we can group the terms in the sum into groups of p identical terms, except for those corresponding to the fixed points $(\{1, \dots, p\}, \emptyset)$, $(\emptyset, \{\{1, \dots, p\}\})$, and $(\emptyset, \{\{1\}, \{2\}, \dots, \{p\}\})$. \square

To finish the calculation of the p -curvature of $\Psi_\zeta^{-1}(S \cdot \mathcal{T}_{X'/S})$, we note that $\nabla_D = D \otimes \text{id} + \zeta^* D$. Thus, in the previous lemma we set $\alpha := D \otimes \text{id}$, and $\beta := \zeta^* D$. Locally, we see that $\beta = \sum_{i=1}^r \langle \zeta(\pi^* d \log m_i), D \rangle \otimes D'_i$, so $\beta_n = \sum_{i=1}^r D^n \cdot \langle \zeta(\pi^* d \log m_i), D \rangle \otimes D'_i$, and in particular $\beta_0, \dots, \beta_{p-1}$ commute pairwise. Also,

$$\begin{aligned} \beta_{p-1} &= \sum_{i=1}^r D^{p-1} \cdot \langle \zeta(\pi^* d \log m_i), D \rangle \otimes D'_i \\ &= \sum_{i=1}^r [\langle \zeta(\pi^* d \log m_i), D^{(p)} \rangle - F_{X/S}^* \langle \pi^* d \log m_i, \pi^* D \rangle] \otimes D'_i = \zeta^*(D^{(p)}) - \pi^* D. \end{aligned}$$

Therefore, $\nabla_D^p = D^{(p)} \otimes \text{id} + \zeta^*(D^{(p)}) - \pi^* D + (\zeta^* D)^p$, and $\nabla_{D^{(p)}} = D^{(p)} \otimes \text{id} + \zeta^*(D^{(p)})$. Hence $\psi_{\pi^* D} = (\zeta^* D)^p - \pi^* D$, as required. \square

We may express this formula geometrically as follows: let $\mathbf{T}_{X/S} = \mathbf{Spec} S \cdot \mathcal{T}_{X/S}$ be the cotangent bundle on X' . Then $\pi_{X/S}^*(\zeta^*)$ gives a map $\mathcal{T}_{X'/S} \rightarrow F_{X'}^* \mathcal{T}_{X'/S}$, which corresponds to a morphism $\phi' : \mathbf{T}_{X'/S}^{(X')} \rightarrow \mathbf{T}_{X'/S}$. Composing with $F_{\mathbf{T}/X'} : \mathbf{T}_{X'/S} \rightarrow \mathbf{T}_{X'/S}^{(X')}$ gives a map $h_\zeta : \mathbf{T}_{X'/S} \rightarrow \mathbf{T}_{X'/S}$, and we see $h_\zeta^*(\pi^* D) = (\zeta^* D)^p$ for $D \in \mathcal{T}_{X/S}$. Thus, letting $\alpha_\zeta := \text{id} - h_\zeta$, we get a commutative

diagram

$$\begin{array}{ccc} HIG(X'/S) & \xrightarrow{\alpha_\zeta * \iota_*} & HIG(X'/S) \\ \downarrow \Psi_\zeta^{-1} & & \downarrow F_{X/S}^* \\ MIC(X/S) & \xrightarrow{\Psi} & F-HIG(X/S), \end{array}$$

and similarly replacing $HIG(X'/S)$, $MIC(X/S)$, and $F-HIG(X/S)$ by $HIG^B(X'/S)$, $MIC^A(X/S)$, and $F-HIG^A(X/S)$, respectively. (We choose this sign for α_ζ so that we may think of it as a perturbation of the identity map.)

Now the restriction $\hat{\alpha}_\zeta$ of α_ζ to the completion $\hat{\mathbf{T}}_{X'/S}$ of $\mathbf{T}_{X'/S}$ over the zero section is an isomorphism, with inverse

$$\hat{\alpha}_\zeta^{-1} = \text{id} + h_\zeta + h_\zeta^2 + \cdots.$$

This allows us to construct a splitting module for $(\tilde{\mathcal{D}}_{X/S})^\wedge := \tilde{\mathcal{D}}_{X/S} \otimes_{S^\wedge \mathcal{T}_{X'/S}} \hat{S}^\wedge \mathcal{T}_{X'/S}$ over $\tilde{\mathcal{Z}}^\wedge := \tilde{\mathcal{Z}} \otimes_{S^\wedge \mathcal{T}_{X'/S}} \hat{S}^\wedge \mathcal{T}_{X'/S} \simeq \mathcal{B}_{X/S} \otimes_{\mathcal{O}_{X'}} \hat{S}^\wedge \mathcal{T}_{X'/S}$, namely $\check{\mathcal{K}}_\zeta^A := \Psi_\zeta^{-1}(\hat{\alpha}_\zeta^{-1})_* \iota_* \tilde{\mathcal{Z}}^\wedge$. By the above diagram we see that the p -curvature on $\check{\mathcal{K}}_\zeta^A \simeq \tilde{\mathcal{Z}}^\wedge \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}}$ is just the natural action of $S^\wedge \mathcal{T}_{X/S}$ on $\tilde{\mathcal{Z}}^\wedge$, so the action of $\hat{S}^\wedge \mathcal{T}_{X'/S}$ on $\tilde{\mathcal{Z}}^\wedge$ and the connection on $\check{\mathcal{K}}_\zeta^A$ extend to a $(\tilde{\mathcal{D}}_{X/S})^\wedge$ -module structure. Also, since $\mathcal{A}_X^{\text{gp}}$ is locally free of rank p^d over $\mathcal{B}_{X/S}$, $\check{\mathcal{K}}_\zeta^A$ is locally free of rank p^d over $\tilde{\mathcal{Z}}^\wedge$, showing that $\check{\mathcal{K}}_\zeta^A$ is indeed a splitting module. However, since $\hat{\alpha}_\zeta^{-1}$ is an isomorphism, we have a canonical isomorphism $\check{\mathcal{K}}_\zeta^A \simeq \tilde{\mathcal{Z}}^\wedge \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}} \simeq \mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_X} F_{X/S}^* \hat{S}^\wedge \mathcal{T}_{X'/S}$.

THEOREM 4.16. *Let $\widehat{MIC}^A(X/S)$ be the category of $(\tilde{\mathcal{D}}_{X/S})^\wedge$ -modules, and let $\widehat{HIG}^B(X'/S)$ be the category of $\tilde{\mathcal{Z}}^\wedge$ -modules. Thus $\widehat{MIC}^A(X/S)$ is equivalent to the category of $\mathcal{A}_X^{\text{gp}}$ -modules with integrable and admissible connections, with an extension of the p -curvature to a $\hat{\mathbf{T}}_{X'/S}$ -Higgs field which commutes with $\mathcal{A}_X^{\text{gp}}$, and $\widehat{HIG}^B(X'/S)$ is equivalent to the $\mathcal{B}_{X/S}$ -modules with $\hat{\mathbf{T}}_{X'/S}$ -Higgs fields which commute with $\mathcal{B}_{X/S}$.*

i) *The functors*

$$\begin{aligned} C_\zeta : \widehat{MIC}^A(X/S) &\rightarrow \widehat{HIG}^B(X'/S), \quad E \mapsto \iota_* \mathcal{H}om_{(\tilde{\mathcal{D}}_{X/S})^\wedge}(\check{\mathcal{K}}_\zeta^A, E), \\ C_\zeta^{-1} : \widehat{HIG}^B(X'/S) &\rightarrow \widehat{MIC}^A(X/S), \quad E' \mapsto \check{\mathcal{K}}_\zeta^A \otimes_{\tilde{\mathcal{Z}}^\wedge} \iota_* E', \end{aligned}$$

are quasi-inverse equivalences of categories.

ii) *Let $E \in \widehat{MIC}^A(X/S)$, and let E' be the corresponding element of $\widehat{HIG}^B(X'/S)$. Then we have isomorphisms*

$$E \simeq \Psi_\zeta^{-1}(\hat{\alpha}_\zeta^{-1})_* E' \simeq \Psi_\zeta^{-1} \hat{\alpha}_\zeta^* E'$$

and

$$\Psi(E) \simeq \iota_* E' \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}}.$$

Here in the second formula, $S^\wedge \mathcal{T}_{X'/S}$ acts on the right hand side via its action on $\iota_ E'$.*

Proof. The first part follows directly from the fact that $\check{\mathcal{K}}_\zeta^A$ is a splitting module. For the second part, since $\Psi_\zeta^{-1} E' \simeq \Psi_\zeta^{-1} S^\wedge \mathcal{T}_{X'/S} \otimes_{S^\wedge \mathcal{T}_{X'/S}} E'$, a base extension gives $\Psi_\zeta^{-1} E' \simeq \Psi_\zeta^{-1} \iota_* \tilde{\mathcal{Z}}^\wedge \otimes_{\tilde{\mathcal{Z}}^\wedge} E'$. Thus, we have

$$E \simeq \Psi_\zeta^{-1}(\hat{\alpha}_\zeta^{-1})_* \iota_* \tilde{\mathcal{Z}}^\wedge \otimes_{\tilde{\mathcal{Z}}^\wedge} \iota_* E' \simeq \Psi_\zeta^{-1}(\hat{\alpha}_\zeta)^* \iota^* \tilde{\mathcal{Z}}^\wedge \otimes_{\tilde{\mathcal{Z}}^\wedge} \iota^* E' \simeq \Psi_\zeta^{-1}(\hat{\alpha}_\zeta)^* E' \simeq \Psi_\zeta^{-1}(\hat{\alpha}_\zeta^{-1})_* E',$$

which proves the first isomorphism. The second follows from the commutative diagram above. \square

Remark 4.17. $\widehat{MIC}^A(X/S)$ is *not* equivalent to $MIC^A(X/S)$. (Indeed, \check{K}_ζ^A is itself a counterexample.) However, $MIC^A(X/S)$ and $HIG^B(X'/S)$ can be regarded as full subcategories of $\widehat{MIC}^A(X/S)$ and $\widehat{HIG}^B(X'/S)$. Similarly to before, when C_ζ and C_ζ^{-1} are restricted to these subcategories, we have isomorphisms $C_\zeta(E) \simeq (\mathcal{K}_\zeta^A \otimes_{\mathcal{A}_X^{\text{gp}}} E)^{\nabla, \gamma}$ and $C_\zeta^{-1}(E') \simeq (\mathcal{K}_\zeta^A \otimes_{\mathcal{B}_{X/S}} E')^\theta$, where $\mathcal{K}_\zeta^A := \Psi_\zeta^{-1}(\alpha_\zeta^{-1})_*(\mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_{X'}} \Gamma.F_{X'/S}^* \Omega_{X'/S}^1) \simeq \mathcal{H}om_{\mathcal{A}_X^{\text{gp}}}(\check{K}_\zeta^A, \mathcal{A}_X^{\text{gp}})$.

Similarly, let $\widehat{MIC}^0(X/S)$ be the category of $\hat{\mathcal{D}}_{X/S} := \mathcal{D}_{X/S} \otimes_{S^*} \hat{S}^* \mathcal{T}_{X'/S}$ -modules with residue ρ satisfying $\rho_D^p = 0$ for all $D \in \mathcal{T}_{X/S}$, and $\widehat{HIG}(X'/S)$ be the category of $\hat{S}^* \mathcal{T}_{X'/S}$ -modules. Also, let $\check{K}_\zeta := \Psi_\zeta^{-1}(\hat{\alpha}_\zeta^{-1})_* \iota_* \hat{S}^* \mathcal{T}_{X'/S}$. Then $C_\zeta^{-1} : E' \mapsto \iota_* E' \otimes_{\hat{S}^* \mathcal{T}_{X'/S}} \check{K}_\zeta$ gives an “almost equivalence” $\widehat{HIG}(X'/S) \rightarrow \widehat{MIC}^0(X/S)$ in the sense of (4.11), with left inverse $C_\zeta : E \mapsto \mathcal{H}om_{\hat{\mathcal{D}}_{X/S}}(\check{K}_\zeta, \iota_* E)$. Yet again, letting $\mathcal{K}_\zeta := \Psi_\zeta^{-1}(\hat{\alpha}_\zeta^{-1})_* \Gamma.\Omega_{X'/S}^1 \simeq \mathcal{H}om_{\mathcal{O}_X}(\check{K}_\zeta, \mathcal{O}_X)$, we get that $C_\zeta(E) \simeq (\mathcal{K}_\zeta \otimes_{\mathcal{O}_X} E)^{\nabla, \gamma}$ for $E \in MIC^0(X/S)$, and $C_\zeta^{-1}(E') \simeq (\mathcal{K}_\zeta \otimes_{\mathcal{O}_{X'}} E')^\theta$ for $E' \in HIG(X'/S)$.

4.5 The Image of the p -curvature Functor

In this section we give an application of the local theory developed in the last section to characterize the essential image of the functor $\Psi : MIC(X/S) \rightarrow F-HIG(X/S)$.

THEOREM 4.18. *Let $X \rightarrow S$ be a smooth morphism of fine log schemes of characteristic p .*

- i) *Let (E, ∇) be a finitely generated $\mathcal{A}_X^{\text{gp}}$ -module with integrable connection on X . Then étale locally on the underlying scheme of X , there exists a \mathcal{B} -Higgs field (E', θ') on X' and an isomorphism $(E, \psi(\nabla)) \simeq (E', \theta') \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}}$, where $\psi(\nabla)$ is the p -curvature of ∇ .*
- ii) *Let (E', θ') be a finitely generated \mathcal{B} -Higgs field on X' . Then étale locally on the underlying scheme on X , there exists an $\mathcal{A}_X^{\text{gp}}$ -module with integrable connection (E, ∇) on X and an isomorphism $(E, \psi(\nabla)) \simeq (E', \theta') \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}}$.*

Proof. We begin by stating the lemma we will use to construct the appropriate étale morphisms.

LEMMA 4.19. *Let $Y \rightarrow X$ be a morphism of noetherian schemes, $\tilde{Y} \rightarrow Y$ a surjective étale morphism, and $Z \rightarrow Y$ a closed immersion such that the composition $Z \rightarrow X$ is finite. Then étale locally on X , there exists a section of $\tilde{Y} \rightarrow Y$ over Z .*

Proof. First, we may replace Y by Z and \tilde{Y} by $\tilde{Z} := \tilde{Y} \times_Y Z$. Thus we may assume that $Y \rightarrow X$ is finite. Suppose that X is the spectrum of a strictly henselian local ring A . Then by Hensel’s lemma, Z splits up as a finite direct sum of schemes Z_i where each Z_i is the spectrum of a strictly henselian local ring. Then each $\tilde{Z}_i := \tilde{Z} \times_Z Z_i$ is étale and surjective over Z_i , and hence admits a section. Thus $\tilde{Z} \rightarrow Z$ also admits a section.

For the general case, let A be a ring endowed with a map $\text{Spec } A \rightarrow X$, and let $F(A)$ denote the set of sections of $\tilde{Z} \rightarrow Z$ over $Z \times_X \text{Spec } A$. Since $\tilde{Z} \rightarrow X$ is of finite presentation, the functor F commutes with direct limits. (We may assume that X is affine, say $\text{Spec } R$. Then Z is also affine, say $\text{Spec } B$, and $Z \times_R A = \text{Spec}(B \otimes_R A)$, and the functor $A \mapsto B \otimes_R A$ commutes with direct limits.) We have seen that $F(A)$ is nonempty if A is the strict henselization of the local ring of X at any point x . But this A is a direct limit of the rings corresponding to étale neighborhoods of x . \square

Now to prove (ii), let $Z \subseteq \mathbf{T}_{X'/S}$ be the support of E' , which is finite over X' since E' is finitely generated (as $\mathcal{B}_{X/S}$ is finitely generated over $\mathcal{A}_{X'}^{\text{gp}}$). Locally, we may choose a splitting ζ of $C_{X/S}$. Then $\alpha_\zeta : \mathbf{T}_{X'/S} \rightarrow \mathbf{T}_{X'/S}$ is surjective étale, so étale locally on X there exists a lifting

$g : Z \rightarrow \mathbf{T}_{X'/S}$ such that $\alpha_\zeta \circ g : Z \rightarrow \mathbf{T}_{X'/S}$ is the closed immersion. Let $(E, \nabla) := \Psi_\zeta^{-1}(\iota_* g_* E')$. Then (E, ∇) has p -curvature $F_{X/S}^*(\alpha_{\zeta*} g_* E') = F_{X/S}^* E'$.

Similarly, for (i), let $Z \subseteq \mathbf{T}_{X'/S}$ be the support of the p -curvature of ∇ , which is again finite over X' . Then as before, we may étale locally choose a splitting ζ such that $\alpha_\zeta : \mathbf{T}_{X'/S} \rightarrow \mathbf{T}_{X'/S}$ has a section g over Z . Then $g_*(E, \psi)$ is a \mathcal{B} -Higgs field compatible with (E, ψ) via α_ζ . Thus, since α_ζ splits the Azumaya algebra $\tilde{D}_{X/S}$, $(E, \psi) \simeq \Psi_\zeta^{-1}(E', \theta')$ for some \mathcal{B} -Higgs field (E', θ') . Therefore, $(E, \psi) \simeq (\alpha_{\zeta*} \iota_* E') \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}}$. \square

Example 4.20. For the case of $E = \mathcal{O}_X$, we can in fact say more: étale locally, (\mathcal{O}_X, ω) is the p -curvature of some integrable connection on \mathcal{O}_X if and only if $\omega = F_{X/S}^* \omega'$ for some $\omega' \in \Omega_{X'/S}^1$. Indeed, for any closed 1-form ω on X , let ∇_ω be the connection on \mathcal{O}_X such that $\nabla_\omega(1) = \omega$. Then the p -curvature of ∇_ω is multiplication by $F_{X/S}^*(\pi^* - C_{X/S})(\omega)$, where $C_{X/S} : Z_{X/S}^1 \rightarrow \Omega_{X'/S}^1$ is the map induced by the Cartier isomorphism.

Conversely, for any $f \in \mathcal{O}_{X'}$, $m \in \mathcal{M}_X^{\text{gp}}$, let $\omega' = f d \log(\pi^* m)$, and let $\theta' = \cdot \omega'$ be the Higgs field on $\mathcal{O}_{X'}$ induced by ω' . Now let $g \in \mathcal{O}_X$ be a root of $g^p - g - F_{X/S}^* f = 0$, which clearly exists étale locally on X . Then since $g = F_{X/S}^*(\pi^* g - f)$,

$$C_{X/S}(g d \log m) = (\pi^* g - f) d \log(\pi^* m) = \pi^*(g d \log m) - f d \log(\pi^* m).$$

Therefore, letting $\omega = g d \log m$, ∇_ω has p -curvature $F_{X/S}^* \omega'$. Since $\pi^* - C_{X/S}$ is additive, this shows that $\pi^* - C_{X/S} : Z_{X/S}^1 \rightarrow \Omega_{X'/S}^1$ is surjective with respect to the étale topology on the underlying scheme of X .

5. De Rham and Higgs Cohomology

Let $\mathcal{X}/\mathcal{S} := (X/S, \tilde{X}'/\tilde{S})$ be as in the previous chapters. The main result of this section is as follows:

THEOREM 5.1. *Let $\ell < p$ be a natural number, let (E, ∇) be an object of $MIC_\ell^A(X/S)$, i.e. an object of $MIC^A(X/S)$ whose p -curvature is nilpotent of level $\leq \ell$, and let (E', θ') be its Cartier transform. Let $m := p - \ell - 1$. Then there is a canonical isomorphism in the derived category of complexes of $\mathcal{B}_{X/\mathcal{S}}$ -modules*

$$\tau_{\leq m}(E \otimes \Omega_{X/S}, \nabla) \simeq \tau_{\leq m}(E' \otimes \Omega_{X'/S}, \theta').$$

Note that if $\ell + \dim(X/S) < p$, this implies that $F_{X/S*}(E \otimes \Omega_{X/S}, \nabla) \simeq (E' \otimes \Omega_{X'/S}, \theta')$ in the derived category.

In fact, we shall make this isomorphism explicit as follows: let

$$\mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,ij}(E) := E \otimes \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes \Omega_{X'/S}^i \otimes \Omega_{X/S}^j.$$

Then for fixed i , the de Rham complex of $E \otimes \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes \Omega_{X'/S}^i$ with the total connection gives a complex

$$\mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,i0}(E) \xrightarrow{d} \mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,i1}(E) \xrightarrow{d} \dots$$

Similarly, for fixed j , the Higgs complex of $(\mathcal{K}_{\mathcal{X}/\mathcal{S}}^A, \psi_{\mathcal{K}^A})$ tensored with the identity on $E \otimes \Omega_{X/S}^j$ gives a complex

$$\mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,0j}(E) \xrightarrow{d'} \mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,1j}(E) \xrightarrow{d'} \dots$$

The differentials d and d' commute and therefore form a double complex $\mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,\bullet,\bullet}(E)$. In fact, we see that d preserves the subcomplex $E \otimes N_n \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes \Omega_{X'/S}^i \otimes \Omega_{X/S}^j$, and that d' maps $E \otimes N_n \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes \Omega_{X'/S}^i \otimes \Omega_{X/S}^j$ to $E \otimes N_{n-1} \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes \Omega_{X'/S}^{i+1} \otimes \Omega_{X/S}^j$. Thus, we get a subcomplex $N_n \mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,\bullet,\bullet}(E)$ defined by

$$N_n \mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,ij}(E) := E \otimes N_{n-i} \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes \Omega_{X'/S}^i \otimes \Omega_{X/S}^j.$$

This double complex looks like:

$$\begin{array}{ccccc} & \uparrow & & \uparrow & & \uparrow \\ E \otimes \mathcal{K}_n^A \otimes \Omega_{X/S}^2 & \xrightarrow{d'} & E \otimes \mathcal{K}_{n-1}^A \otimes \Omega_{X'/S}^1 \otimes \Omega_{X/S}^2 & \xrightarrow{d'} & E \otimes \mathcal{K}_{n-2}^A \otimes \Omega_{X'/S}^2 \otimes \Omega_{X/S}^2 & \longrightarrow \\ \uparrow d & & \uparrow d & & \uparrow d & \\ E \otimes \mathcal{K}_n^A \otimes \Omega_{X/S}^1 & \xrightarrow{d'} & E \otimes \mathcal{K}_{n-1}^A \otimes \Omega_{X'/S}^1 \otimes \Omega_{X/S}^1 & \xrightarrow{d'} & E \otimes \mathcal{K}_{n-2}^A \otimes \Omega_{X'/S}^2 \otimes \Omega_{X/S}^1 & \longrightarrow \\ \uparrow d & & \uparrow d & & \uparrow d & \\ E \otimes \mathcal{K}_n^A & \xrightarrow{d'} & E \otimes \mathcal{K}_{n-1}^A \otimes \Omega_{X'/S}^1 & \xrightarrow{d'} & E \otimes \mathcal{K}_{n-2}^A \otimes \Omega_{X'/S}^2 & \longrightarrow \end{array}$$

We denote by $N_n \mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,\bullet,\bullet}(E)$ the total complex associated to this double complex.

For any natural number n , we have a morphism of complexes

$$E \otimes \Omega_{X/S} \rightarrow E \otimes N_n \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes \Omega_{X/S}$$

via the inclusion $\mathcal{A}_X^{\text{gp}} \rightarrow N_n \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A$, which when composed with d' gives zero. This induces a map $b : E \otimes \Omega_{X/S} \rightarrow N_n \mathcal{K}_{\mathcal{X}/\mathcal{S}}^{A,\bullet,\bullet}(E)$. Similarly, if E has level ℓ , then $E' \simeq (\mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes E)^{\nabla, \gamma} \simeq (N_n \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes E)^{\nabla, \gamma}$ for $n \geq \ell$. Thus, the inclusion $E' \rightarrow N_{n-i} \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes E$ gives a morphism of complexes

$$\tau_{\leq (n-\ell)}(E' \otimes \Omega_{X'/S}) \rightarrow \tau_{\leq (n-\ell)}(E \otimes N_{n-i} \mathcal{K}_{\mathcal{X}/\mathcal{S}}^A \otimes F_{X/S}^* \Omega_{X'/S}),$$

which when composed with d gives zero. This induces a map $a : \tau_{\leq(n-\ell)}(E' \otimes \Omega_{X'/S}) \rightarrow \tau_{\leq(n-\ell)}N_n\mathcal{K}_{X/S}^{\mathcal{A}}(E)$.

If there is a splitting $\zeta : \Omega_{X'/S}^1 \rightarrow F_{X/S*}\Omega_{X/S}^1$ of the Cartier operator $C_{X/S}$, then we may make analogous constructions with $\mathcal{K}_{X/S}^{\mathcal{A}}$ in place of $\mathcal{K}_{X/S}^{\mathcal{A}}$.

THEOREM 5.2. *Let (E, ∇) be an object of $MIC_{\ell}^{\mathcal{A}}(X/S)$.*

- i) *Let $n < p$ be a natural number with $n \geq \ell$, and let $(E', \theta') := C_{X/S}(E, \nabla)$. Then the natural maps*

$$\tau_{\leq(n-\ell)}(E \otimes \Omega_{X/S}, \nabla) \xrightarrow{b} \tau_{\leq(n-\ell)}(N_n\mathcal{K}_{X/S}^{\mathcal{A}}(E)) \xleftarrow{a} \tau_{\leq(n-\ell)}(E' \otimes \Omega_{X'/S}, \theta')$$

are quasi-isomorphisms.

- ii) *Assume that there exists a splitting $\zeta : \Omega_{X'/S}^1 \rightarrow F_{X/S*}\Omega_{X/S}^1$ of $C_{X/S}$. Then the analogs*

$$(E \otimes \Omega_{X/S}, \nabla) \xrightarrow{g} \mathcal{K}_{X/S}^{\mathcal{A}}(E) \xleftarrow{f} (E' \otimes \Omega_{X'/S}, \theta')$$

of a and b are quasi-isomorphisms. In fact, for any integer $n \geq \ell$, the analogs

$$\tau_{\leq(n-\ell)}(E \otimes \Omega_{X/S}, \nabla) \xrightarrow{g_n} \tau_{\leq(n-\ell)}(N_n\mathcal{K}_{X/S}^{\mathcal{A}}(E)) \xleftarrow{f_n} \tau_{\leq(n-\ell)}(E' \otimes \Omega_{X'/S}, \theta')$$

of a and b are quasi-isomorphisms.

Note that (5.1) follows easily from the case $n = p - 1$ in the first part.

Example 5.3. Let $(E, \nabla) := (\mathcal{A}_X^{\text{gp}}, d)$, so that $(E', \theta') = (\mathcal{B}_{X/S}, 0)$. Then for $m \in \mathcal{M}_X^{\text{gp}}$, the image of $d \log m \in \Omega_{X/S}^1$ under b is $1 \otimes d \log m \in \mathcal{K}^{\mathcal{A}} \otimes \Omega_{X/S}^1$; similarly, the image of $\pi^* d \log m \in \Omega_{X'/S}^1$ under a is $1 \otimes \pi^* d \log m \in \mathcal{K}^{\mathcal{A}} \otimes \Omega_{X'/S}^1$. However, if we have a lifting $\tilde{\pi}$ of $\pi_{X/S}$ and a lifting $\tilde{m} \in \mathcal{M}_{\tilde{X}}^{\text{gp}}$ of m , then

$$(d \oplus d')\beta_{\tilde{\pi}}(\tilde{m}) = 1 \otimes d \log m - 1 \otimes \pi^* d \log m \in (\mathcal{K}^{\mathcal{A}} \otimes \Omega_{X/S}^1) \oplus (\mathcal{K}^{\mathcal{A}} \otimes \Omega_{X'/S}^1).$$

Hence $a(\pi^* d \log m)$ and $b(d \log m)$ have the same image in $\mathcal{H}^1\mathcal{K}_{X/S}^{\mathcal{A}}$, so the isomorphism above is compatible with the standard Cartier isomorphism. In general, if we only have a local lifting \tilde{F} of $F_{X/S}$, then the difference is locally the image of $g + \sigma_{\tilde{F}}(1 \otimes \pi^* d \log m)$, where we construct $g \in \mathcal{O}_X$ by taking a local lifting $\tilde{m}' \in \mathcal{M}_{\tilde{X}'}^{\text{gp}}$ of $\pi^* m$ and letting g satisfy $1 + [p]g = \alpha_{\tilde{X}}(\tilde{F}^*\tilde{m}' - p\tilde{m})$.

To prove (5.2), we begin with the case $(E, \nabla) = (\mathcal{A}_X^{\text{gp}}, d)$.

LEMMA 5.4. *Assume that ζ is a splitting of $C_{X/S}$.*

- i) *The Higgs complex $(\mathcal{K}_{X/S}^{\mathcal{A}} \otimes F_{X/S}^*\Omega_{X'/S}, \psi)$ is a resolution of $\mathcal{A}_X^{\text{gp}}$. In fact, for any $n \geq 0$,*

$$N_n\mathcal{K}_{X/S}^{\mathcal{A}} \xrightarrow{\psi} N_{n-1}\mathcal{K}_{X/S}^{\mathcal{A}} \otimes \Omega_{X'/S}^1 \xrightarrow{\psi} N_{n-2}\mathcal{K}_{X/S}^{\mathcal{A}} \otimes \Omega_{X'/S}^2 \xrightarrow{\psi} \dots$$

is also a resolution of $\mathcal{A}_X^{\text{gp}}$.

- ii) *The de Rham complex $(\mathcal{K}_{X/S}^{\mathcal{A}} \otimes \Omega_{X/S}, \nabla)$ is a resolution of $\mathcal{B}_{X/S}$.*

Proof. In (i), the statement for $(\mathcal{K}_{X/S}^{\mathcal{A}} \otimes \Omega_{X'/S}, \psi)$ follows from the statement for $(N_{n-1}\mathcal{K}_{X/S}^{\mathcal{A}} \otimes \Omega_{X'/S}, \psi)$ by taking direct limits. Since the latter complex is just $\mathcal{A}_X^{\text{gp}}$ for $n = 0$, it suffices to show that the successive quotients are exact. However, these quotients are equivalent to

$$\Gamma_n(\mathcal{A}_X^{\text{gp}} \otimes \Omega_{X'/S}^1) \rightarrow \Gamma_{n-1}(\mathcal{A}_X^{\text{gp}} \otimes \Omega_{X'/S}^1) \otimes \Omega_{X'/S}^1 \rightarrow \Gamma_{n-2}(\mathcal{A}_X^{\text{gp}} \otimes \Omega_{X'/S}^1) \otimes \Omega_{X'/S}^2 \rightarrow \dots,$$

with the maps given by tensoring the maps in the divided power de Rham complex of $\Omega_{X'/S}^1$ by $\mathcal{A}_X^{\text{gp}}$. Thus, the result follows by the divided power Poincaré lemma.

To prove (ii), we note that the connection on $\mathcal{K}_\zeta^{\mathcal{A}}$ preserves the filtration N ., and we consider the spectral sequence of the filtered complex $(N\mathcal{K}_\zeta^{\mathcal{A}} \otimes \Omega_{X/S}, \nabla)$. In order to be compatible with standard notation, let us denote $N^i := N_{-i}$. We begin with an indexed version of [Ogu04, 5.1.1], which in fact implies this version by applying it in each degree.

LEMMA 5.5. *Let $(E, \nabla) \in MIC^{\mathcal{A}}(X/S)$, and let $\psi : E \rightarrow E \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}^1$ be the p -curvature of ∇ . Suppose we have a horizontal filtration N on E such that ψ vanishes on $\text{Gr}_N^i E$ for each E . Then in the spectral sequence of the filtered complex $(N^* E \otimes \Omega_{X/S}, \nabla)$, there is a commutative diagram*

$$\begin{array}{ccc} E_1^{i,j} & \xrightarrow{d_1^{i,j}} & E_1^{i+1,j} \\ \downarrow \sim & & \downarrow \sim \\ \text{Gr}_N^i(E)^\nabla \otimes \Omega_{X'/S}^{i+j} & \xrightarrow{-\bar{\psi}} & \text{Gr}_N^{i+1}(E)^\nabla \otimes \Omega_{X'/S}^{i+j+1}, \end{array}$$

where $\bar{\psi}$ is the map induced by ψ .

Proof. For any fixed i, j , by replacing E with $E/N^k E$ for k sufficiently large, we may assume E is locally nilpotent. Then given a local lifting ζ of $C_{X/S}$, by (4.16), (E, ∇) is in the essential image of Ψ_ζ^{-1} ; say $(E, \nabla) \simeq \Psi_\zeta^{-1}(E', \theta')$. Also, the filtration on E descends to a filtration on E' . Now we have $\psi = \alpha_{\zeta*} \theta' \otimes \text{id} : E' \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}} \rightarrow E' \otimes_{\mathcal{B}_{X/S}} \mathcal{A}_X^{\text{gp}} \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}^1$. However, since E' is locally nilpotent and $\hat{\alpha}_\zeta$ is an isomorphism, we get that θ' vanishes on $N^i E'$.

Now by definition, $\nabla = d_{\mathcal{A}} + (\text{id} \otimes \zeta) \circ \theta'$; however, $(\text{id} \otimes \zeta) \circ \theta'$ maps $N^i(E \otimes \Omega_{X/S}^{i+j})$ to $N^{i+1}(E \otimes \Omega_{X/S}^{i+j+1})$. Therefore, $E_1^{i,j} \simeq \mathcal{H}^{i+j}(\text{Gr}_N^i E' \otimes \mathcal{A}_X^{\text{gp}} \otimes \Omega_{X/S}, d_{\mathcal{A}})$, and $d_1^{i,j}$ is the map induced by $(\text{id} \otimes \zeta) \circ \theta'$. However, since each term of the complex $(\mathcal{A}_X^{\text{gp}} \otimes \Omega_{X/S}, d)$ is locally free over $\mathcal{A}_X^{\text{gp}}$ and therefore over $\mathcal{B}_{X/S}$, and the differential d is $\mathcal{B}_{X/S}$ -linear, from (2.6) we conclude

$$E_1^{i,j} \simeq \text{Gr}_N^i(E') \otimes \Omega_{X'/S}^{i+j} \simeq E_1^{0,j} \otimes \Omega_{X'/S}^{i+j} \simeq \text{Gr}_N^i(E)^\nabla \otimes \Omega_{X'/S}^{i+j}.$$

This gives the vertical isomorphisms. Also, since $C_{X/S} \circ \zeta = \text{id}$, it is easy to see that the map $\text{Gr}_N^i(E') \otimes \Omega_{X'/S}^{i+j} \rightarrow \text{Gr}_N^{i+1}(E') \otimes \Omega_{X'/S}^{i+j+1}$ induced by $(\text{id} \otimes \zeta) \circ \theta'$ is just $\bar{\theta}'$. However, $\psi = \alpha_{\zeta*} \theta' \otimes \text{id}$, while h_ζ maps $N^i E' \otimes \mathcal{A}_X^{\text{gp}} \otimes \Omega_{X'/S}^{i+j}$ to $N^{i+p} E' \otimes \mathcal{A}_X^{\text{gp}} \otimes \Omega_{X'/S}^{i+j+1}$, so $\bar{\theta}' = -\bar{\psi}$. \square

In our situation with $E = \mathcal{K}_\zeta^{\mathcal{A}} \simeq \mathcal{A}_X^{\text{gp}} \otimes \Gamma_i \Omega_{X'/S}^1$, we have $\text{Gr}_N^{-i}(E) \simeq \mathcal{A}_X^{\text{gp}} \otimes \Gamma_i \Omega_{X'/S}^1$ with connection $d_{\mathcal{A}}$, so $E_1^{-i,j} \simeq \mathcal{B}_{X/S} \otimes \Gamma_i \Omega_{X'/S}^1 \otimes \Omega_{X'/S}^{j-i}$. Also, the p -curvature of $\mathcal{K}_\zeta^{\mathcal{A}}$ is exactly the map we get by tensoring the standard differential d in the divided power de Rham complex of $\Gamma_i \Omega_{X'/S}^1$ with $\mathcal{A}_X^{\text{gp}}$, so that we get a commutative diagram

$$\begin{array}{ccc} E_1^{-i,j} & \xrightarrow{d_1^{-i,j}} & E_1^{-i+1,j} \\ \downarrow \sim & & \downarrow \sim \\ \mathcal{B}_{X/S} \otimes \Gamma_i \Omega_{X'/S}^1 \otimes \Omega_{X'/S}^{j-i} & \xrightarrow{-\text{id} \otimes d} & \mathcal{B}_{X/S} \otimes \Gamma_{i-1} \Omega_{X'/S}^1 \otimes \Omega_{X'/S}^{j-i+1}. \end{array}$$

Thus, by the divided power Poincaré lemma, we get $E_2^{i,j} = 0$ unless $i = j = 0$. Hence the cohomology of $(\mathcal{K}_\zeta^{\mathcal{A}} \otimes \Omega_{X/S}, \nabla)$ vanishes in positive degree, and we have an isomorphism

$$\mathcal{H}^0(\mathcal{K}_\zeta^{\mathcal{A}} \otimes \Omega_{X/S}, \nabla) \simeq E_2^{0,0} \simeq \mathcal{B}_{X/S}.$$

\square

LEMMA 5.6. Let $(E, \nabla) \in MIC^{\mathcal{A}}(X/S)$, and let $(E', \theta') := C_{\zeta}(E, \nabla) \in HIG^{\mathcal{B}}(X'/S)$ be its Cartier transform.

- i) The Higgs complex $(E \otimes \mathcal{K}_{\zeta}^{\mathcal{A}} \otimes \Omega_{X'/S}^{\bullet}, \psi_{\mathcal{K}})$ is a resolution of E . In fact, for any natural number n , the complex

$$E \otimes N_n \mathcal{K}_{\zeta}^{\mathcal{A}} \xrightarrow{\psi_{\mathcal{K}}} E \otimes N_{n-1} \mathcal{K}_{\zeta}^{\mathcal{A}} \otimes \Omega_{X'/S}^1 \xrightarrow{\psi_{\mathcal{K}}} E \otimes N_{n-2} \mathcal{K}_{\zeta}^{\mathcal{A}} \otimes \Omega_{X'/S}^2 \xrightarrow{\psi_{\mathcal{K}}} \dots$$

is a resolution of E .

- ii) The de Rham complex $(E \otimes \mathcal{K}_{\zeta}^{\mathcal{A}} \otimes \Omega_{X/S}^{\bullet}, \nabla_{tot})$ is a resolution of E' .

Proof. Note that the statement for (i) does not involve the connection on E . Thus, since each term in the complex $\mathcal{K}_{\zeta}^{\mathcal{A}} \otimes \Omega_{X'/S}^{\bullet}$ is a locally free $\mathcal{A}_X^{\text{gp}}$ -module, and the differential is $\mathcal{A}_X^{\text{gp}}$ -linear, the result follows from (5.4.i). The proof for $E \otimes N_{n-1} \mathcal{K}_{\zeta}^{\mathcal{A}} \otimes \Omega_{X'/S}^{\bullet}$ is similar.

To prove (ii), recall that since E is locally nilpotent, $E \simeq (E' \otimes_{\mathcal{B}_{X/S}} \mathcal{K}_{\zeta}^{\mathcal{A}})^{\theta}$. We thus have a map $E \otimes_{\mathcal{A}_X^{\text{gp}}} \mathcal{K}_{\zeta}^{\mathcal{A}} \rightarrow E' \otimes_{\mathcal{B}_{X/S}} \mathcal{K}_{\zeta}^{\mathcal{A}}$ induced by the multiplication morphism $\mathcal{K}_{\zeta}^{\mathcal{A}} \otimes \mathcal{K}_{\zeta}^{\mathcal{A}} \rightarrow \mathcal{K}_{\zeta}^{\mathcal{A}}$, which is an isomorphism by a result in the appendix to [OV]. This isomorphism is horizontal since the connection on $\mathcal{K}_{\zeta}^{\mathcal{A}}$ satisfies the Leibniz rule and since the connection on $E \simeq (E' \otimes \mathcal{K}_{\zeta}^{\mathcal{A}})^{\theta}$ comes from the action of ∇ on $\mathcal{K}_{\zeta}^{\mathcal{A}}$. Therefore, the de Rham complex $(E \otimes \mathcal{K}_{\zeta}^{\mathcal{A}} \otimes \Omega_{X/S}^{\bullet}, \nabla_{tot})$ comes from tensoring the de Rham complex $(\mathcal{K}_{\zeta}^{\mathcal{A}} \otimes \Omega_{X/S}^{\bullet}, \nabla_{tot})$ with E' over $\mathcal{B}_{X/S}$. Since each term in the latter complex is a locally free $\mathcal{B}_{X/S}$ -module (using the fact that $\mathcal{A}_X^{\text{gp}}$ itself is), and the differential is $\mathcal{B}_{X/S}$ -linear, the desired result follows from (5.4.ii). \square

Proof of 5.2. Clearly, (i) follows from (ii) since locally we may choose a splitting ζ of $C_{X/S}$, and then $N_n \mathcal{K}_{X/S}^{\mathcal{A}} \simeq N_n \mathcal{K}_{\zeta}^{\mathcal{A}}$ for $n < p$.

To prove (ii), observe that we have a commutative diagram

$$\begin{array}{ccc} \tau_{\leq(n-\ell)}(E \otimes \Omega_{X/S}^{\bullet}, \nabla) & & \\ \downarrow g_n & \searrow g & \\ \tau_{\leq(n-\ell)} N_n \mathcal{K}_{\zeta}^{\mathcal{A}}(E) & \xrightarrow{c} & \tau_{\leq(n-\ell)} \mathcal{K}_{\zeta}^{\mathcal{A}}(E) \\ \uparrow f_n & \nearrow f & \\ \tau_{\leq(n-\ell)}(E' \otimes \Omega_{X'/S}^{\bullet}, \theta') & & \end{array}$$

By the previous lemma, the rows of $N_n \mathcal{K}_{\zeta}^{\mathcal{A}}(E)$ and of $\mathcal{K}_{\zeta}^{\mathcal{A}}(E)$ are resolutions of $E \otimes \Omega_{X/S}^i$; therefore, g_n and g are quasi-isomorphisms. Similarly, the columns of $\mathcal{K}_{\zeta}^{\mathcal{A}}(E)$ are resolutions of $E' \otimes \Omega_{X'/S}^i$, which implies that f is a quasi-isomorphism. From the above diagram, it follows that c and f_n are also quasi-isomorphisms. \square

If $E \in MIC_{\ell}(X/S)$ for $\ell < p$, and we apply this result to $E \otimes_{\mathcal{O}_X} \mathcal{A}_X^{\text{gp}} \in MIC_{\ell}^{\mathcal{A}}(X/S)$, then take the degree zero parts, we get the following.

COROLLARY 5.7. Let $(E, \nabla) \in MIC_{\ell}(X/S)$ for $\ell < p$, and let $E' := (\mathcal{K}_{X/S} \otimes_{\mathcal{O}_X} E)^{\nabla, \gamma}$, with Higgs field $\theta' := \psi_{\mathcal{K}}$. (Thus $(E', \theta') \simeq C_{X/S}(E, \nabla)$ in the case that $(E, \nabla) \in MIC_{\ell}^0(X/S)$.) Let $m := p - \ell - 1$. Then there is a quasi-isomorphism in the derived category of complexes of $\mathcal{O}_{X'}$ -modules

$$\tau_{\leq m}(E \otimes_{\mathcal{O}_X} \Omega_{X/S}^{\bullet}, \nabla) \simeq \tau_{\leq m}(E' \otimes_{\mathcal{O}_{X'}} \Omega_{X'/S}^{\bullet}, \theta').$$

Appendix A. A Brief Proof of the F -linearity of p -curvature

Fix the following notation: let $f : X \rightarrow S$ be a morphism of fine log schemes of characteristic p . Let Y be the logarithmic formal neighborhood of the diagonal in $X \times_S X$, with the exact closed immersion $\Delta : X \rightarrow Y$ and projections $p_1, p_2 : Y \rightarrow X$. Let $\mathcal{P}_{X/S} = \mathcal{O}_Y$, let J denote the ideal of Δ , and let $\mathcal{P}_{X/S}^n = \mathcal{P}_{X/S}/J^{n+1}$ denote the structure sheaf of the n th log infinitesimal neighborhood of Δ . Similarly, let $\mathcal{D}_{X/S}(1)$ denote the structure sheaf of the log divided power envelope of Δ , with PD ideal \bar{J} , and $\mathcal{D}_{X/S}^n(1) = \mathcal{D}_{X/S}(1)/\bar{J}^{[n+1]}$ the structure sheaf of the n th log infinitesimal divided power envelope. We consider $\mathcal{P}_{X/S}$, $\mathcal{P}_{X/S}^n$, $\mathcal{D}_{X/S}(1)$, and $\mathcal{D}_{X/S}^n(1)$ as \mathcal{O}_X -algebras via p_1^* , i.e. by multiplication on the left.

For m a section of $\mathcal{M}_X^{\text{gp}}$, we define $\eta_m := \alpha_Y(p_2^*m - p_1^*m) - 1 \in J$, and $\zeta_m := \log(1 + \eta_m) = \eta_m - \eta_m^{[2]} + 2!\eta_m^{[3]} - \dots + (p-1)!\eta_m^{[p]} \in \bar{J}$. (The reader who is unfamiliar with log geometry can skip to Lemma A.6, and replace ζ_m by $\xi_x := 1 \otimes x - x \otimes 1 \in \bar{J}$ for sections x of \mathcal{O}_X .)

We begin with a logarithmic generalization of a result which is well-known in the nonlogarithmic case.

THEOREM A.1. *Let $D : \mathcal{P}^1 \rightarrow \mathcal{O}_X$ be a differential operator of order ≤ 1 . Then $D^p : \mathcal{P}^p \rightarrow \mathcal{O}_X$ is also a differential operator of order ≤ 1 .*

In the case of trivial log structure, we prove this by showing that $[D^p, a]$ is a differential operator of order ≤ 0 for each $a \in \mathcal{O}_X$. We will essentially adapt this proof to the logarithmic case by substituting $a = \alpha(m)$ for $m \in \mathcal{M}_X$ and dividing each step by $\alpha(m)$. This leads to the following construction:

DEFINITION A.2. For $m \in \mathcal{M}_X^{\text{gp}}$, and $\phi : \mathcal{P} \rightarrow \mathcal{O}_X$, we define $\phi_m : \mathcal{P} \rightarrow \mathcal{O}_X$ by $\phi_m(\tau) = \phi(\tau(1 + \eta_m))$ for each $\tau \in \mathcal{P}$.

Note that for $m \in \mathcal{M}_X$, since $\alpha(m)(1 + \eta_m) = p_2^*(\alpha(m))$, we have $\alpha(m)\phi_m = \phi \circ \alpha(m)$. Thus, we may view ϕ_m as conjugation of ϕ by $\alpha(m)$.

LEMMA A.3. *The collection of maps $\text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \rightarrow \text{Diff}(\mathcal{O}_X, \mathcal{O}_X)$ defined by $\phi \mapsto \phi_m$ induces a group homomorphism $\mathcal{M}_X^{\text{gp}} \rightarrow \text{Aut}_{\mathcal{O}_X}(\text{Diff}(\mathcal{O}_X, \mathcal{O}_X))$.*

Proof. Since $(1 + \eta_m)(1 + \eta_{m'}) = 1 + \eta_{m+m'}$, we see that $(\phi_m)_{m'} = \phi_{m+m'}$. We also have $\eta_0 = 0$, so $\phi_0 = \phi$. Since $\phi \mapsto \phi_m$ is clearly \mathcal{O}_X -linear, all we have left to prove is that $(\phi \circ \psi)_m = \phi_m \circ \psi_m$.

Thus, suppose $\tau \in \mathcal{P}$. By definition, we calculate $(\phi \circ \psi)(\tau(1 + \eta_m))$ by first taking $\delta(\tau(1 + \eta_m)) = \delta(\tau)\delta(1 + \eta_m)$. However, $\delta(1 + \eta_m) = (1 + \eta_m) \otimes (1 + \eta_m)$, so $\delta(\tau(1 + \eta_m))$ gets mapped by $\text{id} \otimes \psi$ to $(1 + \eta_m) \cdot (\text{id} \otimes \psi_m)(\delta(\tau))$. Applying ϕ , we get $(\phi_m \circ \psi_m)(\tau)$, as desired. \square

In the case of a scheme with trivial log structure, we have the result that a differential operator $\phi : \mathcal{P} \rightarrow \mathcal{O}_X$ is of order $\leq k$ if and only if $[\phi, a]$ is of order $\leq (k-1)$ for each $a \in \mathcal{O}_X$. The following is a logarithmic analogue of this fact.

LEMMA A.4. *A differential operator $\phi : \mathcal{P} \rightarrow \mathcal{O}_X$ (of finite order) is of order $\leq k$ if and only if $\phi_m - \phi$ is of order $\leq (k-1)$ for each $m \in \mathcal{M}_X^{\text{gp}}$.*

Proof. Suppose ϕ is of order $\leq \ell$, so that ϕ induces a map $\mathcal{P}^\ell \rightarrow \mathcal{O}_X$. Then for $\tau \in \mathcal{P}^\ell$, $(\phi_m - \phi)(\tau) = \phi(\tau\eta_m)$; thus, $\phi_m - \phi$ is of order $\leq (k-1)$ if and only if $\phi(\tau\eta_m) = 0$ whenever $\tau \in J^k$. However, since $\{\eta_m : m \in \mathcal{M}_X\}$ generates J as an \mathcal{O}_X -module in \mathcal{P}^ℓ , this condition is sufficient (and obviously necessary) to get $\phi(J^{k+1}) = 0$. \square

Proof of Theorem A.1. First consider the case in which $D(1) = 0$. By hypothesis together with the last lemma, we see that $D_m - D = a$ for some $a \in \mathcal{O}_X$. Therefore, $(D^p)_m = (D_m)^p = (D + a)^p$. However, a straightforward induction, using the formula $(D + a) \circ x = (D + a)(x) + xD$ for $x \in \mathcal{O}_X$, shows that

$$(D + a)^k = \sum_{i=0}^k \binom{k}{i} b_i D^{k-i}$$

for every k , where $b_i = (D + a)^i(1)$. In particular, for $k = p$, we get $(D + a)^p = D^p + b_p$, so $(D^p)_m - D^p = b_p$ is a differential operator of order ≤ 0 .

Now in the general case, the previous paragraph shows that $(D - D(1))^p$ is a differential operator of order ≤ 1 . But using the above formula again, $D^p = (D - D(1))^p + b'_p$ for some $b'_p \in \mathcal{O}_X$, so D^p is also a differential operator of order ≤ 1 . \square

COROLLARY A.5. *If $D = (\delta, \partial) : (\mathcal{M}_X, \mathcal{O}_X) \rightarrow \mathcal{O}_X$ is a log derivation over S , then so is $D^{(p)} := (\partial^{p-1} \circ \delta + F_X^* \circ \delta, \partial^{(p)})$.*

Proof. Corresponding to the given log derivation, we have a PD differential operator D of order ≤ 1 with $D(1) = 0$. Then by Theorem A.1, $(D^p)^\flat : \mathcal{P} \rightarrow \mathcal{O}_X$ is a differential operator of order ≤ 1 , so $(D^p)^\flat : \mathcal{P}/J^2 \rightarrow \mathcal{O}_X$ restricts to a log derivation $J/J^2 \simeq \Omega_{X/S}^1 \rightarrow \mathcal{O}_X$. We claim that this log derivation is exactly $\partial^{p-1} \circ \delta + F_X^* \circ \delta$ on \mathcal{M}_X and $\partial^{(p)}$ on \mathcal{O}_X .

The latter statement follows exactly as in the case of trivial log structure. Now by the identification of J/J^2 with $\Omega_{X/S}^1$, we see the log derivation sends $m \in \mathcal{M}_X$ to $D^p(\eta_m)$. However, $\eta_m \equiv \zeta_m + \zeta_m^{[2]} + \cdots + \zeta_m^{[p]} \pmod{\bar{J}^{[p+1]}}$. On the other hand, for $1 < k < p$, $\zeta_m^{[k]}$ is in the image of the natural map $J^2 \subseteq \mathcal{P} \rightarrow \mathcal{D}(1)$, so $D^p(\zeta_m^{[k]}) = 0$. Therefore, $D^p(\eta_m) = D^p(\zeta_m) + D^p(\zeta_m^{[p]})$.

We calculate $D^p(\zeta_m)$ as $(D^{p-1} \circ D)(\zeta_m)$. We have $\delta(\zeta_m) = \zeta_m \otimes 1 + 1 \otimes \zeta_m$, which is mapped by $\text{id} \otimes D$ to $p_2^*(\delta m)$ since $D(1) = 0$ and $D(\zeta_m) = D(\eta_m) = \delta m$. Now applying D^{p-1} gives $\partial^{p-1}(\delta m)$. The following lemma will give that $D^p(\zeta_m^{[p]}) = D(\zeta_m)^p = (\delta m)^p$. \square

LEMMA A.6. *Let $\phi, \psi : \mathcal{D}(1) \rightarrow \mathcal{O}_X$ be PD differential operators of order $\leq k, \ell$, respectively. Then for any section m of \mathcal{M}_X , we have $(\phi \circ \psi)(\zeta_m^{[k+\ell]}) = \phi(\zeta_m^{[k]})\psi(\zeta_m^{[\ell]})$.*

Proof. By definition, we calculate $\phi \circ \psi$ by first taking $\delta^{k,\ell}(\zeta_m^{[k+\ell]}) = \sum_{j=0}^{k+\ell} \zeta_m^{[j]} \otimes \zeta_m^{[k+\ell-j]} = \zeta_m^{[k]} \otimes \zeta_m^{[\ell]}$ in $\mathcal{D}^k(1) \otimes \mathcal{D}^\ell(1)$. We now apply $\text{id} \otimes \psi$, which gives $\zeta_m^{[k]} p_2^* \psi(\eta_m^{[\ell]})$. However, since $\zeta_m^{[k]} \in \bar{J}^{[k]}$ and $p_2^* \psi(\zeta_m^{[\ell]}) - p_1^* \psi(\zeta_m^{[\ell]}) \in \bar{J}$, we see that $\eta_m^{[k]} p_2^* \psi(\zeta_m^{[\ell]}) = \psi(\zeta_m^{[\ell]}) \zeta_m^{[k]}$ in $\mathcal{D}^{k+\ell}(1)$. Thus, applying ϕ gives $\psi(\zeta_m^{[\ell]}) \phi(\zeta_m^{[k]})$, as desired. \square

We will now show the linearity properties of the p -curvature by showing directly that a construction of Mochizuki agrees with the standard definition of the p -curvature. Mochizuki's construction begins with the following observation:

PROPOSITION A.7. *Let $(D(1), \bar{I}, \gamma)$ denote the logarithmic PD envelope of the diagonal in Y (the logarithmic formal neighborhood of the diagonal in $X \times_S X$), and I the ideal of the diagonal in Y . Then there is a unique isomorphism*

$$\alpha : F_{X/S}^* \Omega_{X'/S}^1 \rightarrow \bar{I}/(\bar{I}^{[p+1]} + I\mathcal{O}_{D(1)})$$

such that for any $\xi \in I$ with image $\omega \in I/I^2 \simeq \Omega_{X/S}^1$,

$$\alpha(1 \otimes \pi^* \omega) = \xi^{[p]}.$$

Proof. First, the corresponding map $\Omega_{X/S}^1 \rightarrow F_{X*}[\bar{I}/(\bar{I}^{[p+1]} + I\mathcal{O}_{D(1)})]$ is additive since

$$(\xi + \tau)^{[p]} = \xi^{[p]} + \tau^{[p]} + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} \xi^{[p-i]} \tau^{[i]},$$

where the last term is in $I\mathcal{O}_{D(1)}$. It is also \mathcal{O}_X -linear since

$$(a\xi)^{[p]} = a^p \xi^{[p]}.$$

Finally, for $\xi, \tau \in I$,

$$(\xi\tau)^{[p]} = \xi^p \tau^{[p]} \in I\mathcal{O}_{D(1)},$$

so the map annihilates I^2 . We thus get a well-defined map

$$F_{X/S}^* \Omega_{X'/S}^1 \simeq F_X^* \Omega_{X/S}^1 \rightarrow \bar{I}/(\bar{I}^{[p+1]} + I\mathcal{O}_{D(1)}).$$

To see this is an isomorphism, we work locally; thus, assume we have a logarithmic system of coordinates $m_1, \dots, m_r \in \mathcal{M}_X$. Then $\bar{I}/(\bar{I}^{[p+1]} + I\mathcal{O}_{D(1)})$ has basis $\eta^{[p\epsilon_1]}, \dots, \eta^{[p\epsilon_r]}$, which is the image under α of the basis $1 \otimes \pi^*(d \log m_1), \dots, 1 \otimes \pi^*(d \log m_r)$ of $F_{X/S}^* \Omega_{X'/S}^1$. \square

Now let E be a crystal of $\mathcal{O}_{X/S}$ -modules, and let $p_1, p_2 : D(1) \rightarrow X$ be the canonical projections. Then we get an isomorphism

$$\epsilon : p_2^* E \xrightarrow{\theta_{p_2}} E_{D(1)} \xrightarrow{\theta_{p_1}^{-1}} p_1^* E.$$

Now for $e \in E$, $\epsilon(p_2^* e) - p_1^* e \in E \otimes_{\mathcal{O}_X} \bar{I}$, so this induces a map

$$\psi : E \rightarrow E \otimes_{\mathcal{O}_X} [\bar{I}/(\bar{I}^{[p+1]} + I\mathcal{O}_{D(1)})] \simeq E \otimes_{\mathcal{O}_X} F_{X/S}^* \Omega_{X'/S}^1.$$

THEOREM A.8. *Let $D \in \mathcal{T}_{X/S}$. Then*

$$\psi_{\pi^* D} : E \xrightarrow{\psi} E \otimes_{\mathcal{O}_X} F_{X/S}^* \Omega_{X'/S}^1 \xrightarrow{\text{id} \otimes F_{X/S}^* \pi^* D} E$$

is equal to $\nabla_D^p - \nabla_{D^{(p)}}$.

Proof. First, $D^p - D^{(p)} : \mathcal{O}_{D(1)} \rightarrow \mathcal{O}_X$ annihilates $I\mathcal{O}_{D(1)}$ by the definition of $D^{(p)}$, and it annihilates $\bar{I}^{[p+1]}$ since it is a PD differential operator of order $\leq p$. Therefore, $D^p - D^{(p)}$ induces a well-defined map

$$F_{X/S}^* \Omega_{X'/S}^1 \simeq \bar{I}/(\bar{I}^{[p+1]} + I\mathcal{O}_{D(1)}) \rightarrow \mathcal{O}_X.$$

We claim that in fact, this map agrees with $\text{id} \otimes F_{X/S}^* \pi^* D$. It suffices to check this for $1 \otimes \pi^* \omega$ for $\omega \in \Omega_{X/S}^1$. However, taking a preimage $\xi \in I$ of ω , we get $\alpha(1 \otimes \pi^* \omega) = \xi^{[p]}$, which is mapped by D^p to $(D\omega)^p = F_X^*(D\omega)$ and by $D^{(p)}$ to zero. On the other hand, $\text{id} \otimes F_{X/S}^* \pi^* D$ maps $1 \otimes \pi^* \omega$ to $F_X^*(D\omega)$ also.

Therefore, we may rewrite $\psi_{\pi^* D}$ as

$$E \xrightarrow{\bar{\psi}} E \otimes_{\mathcal{O}_X} \bar{I} \rightarrow E \otimes_{\mathcal{O}_X} (\bar{I}/(\bar{I}^{[p+1]} + I\mathcal{O}_{D(1)})) \xrightarrow{\text{id} \otimes (D^p - D^{(p)})} E,$$

where $\bar{\psi} : E \rightarrow E \otimes_{\mathcal{O}_X} \bar{I}$ maps e to $\epsilon(p_2^* e) - p_1^* e$. However, this is exactly $\nabla_{D^p - D^{(p)}}$. \square

The immediate corollary of this is:

COROLLARY A.9. *If $\nabla : E \rightarrow E \otimes_{\mathcal{O}_X} \Omega_{X/S}^1$ is an integrable connection, then the p -curvature map $\psi(\nabla) : \mathcal{T}_{X/S} \rightarrow F_{X*} \mathcal{E}nd_{\mathcal{O}_X}(E)$, $D \mapsto \nabla_D^p - \nabla_{D^{(p)}}$, is F -linear.*

A more concrete proof of the linearity starts with the following criterion:

LEMMA A.10. *Let $\phi : \mathcal{D}^p(1) \rightarrow \mathcal{O}_X$ be a PD differential operator of order $\leq p$. Then ϕ is of order ≤ 1 if and only if $\phi^b : \mathcal{P}^p \rightarrow \mathcal{O}_X$ is a differential operator of order ≤ 1 , and $\phi(\zeta_m^{[p]}) = 0$ for every $m \in \mathcal{M}_X$.*

Proof. The forward implication is trivial. For the reverse implication, since the statement is local, we may choose a logarithmic system of coordinates $m_1, \dots, m_r \in \mathcal{M}_X$, and let $\zeta^{[k]}$ ($k \in \mathbb{N}^r$) be the corresponding basis for $\mathcal{D}(1)$. Let $\epsilon_1, \dots, \epsilon_r$ denote the canonical basis of \mathbb{N}^r . Since ϕ is of order $\leq p$, $\phi(\zeta^{[k]}) = 0$ for $|k| > p$. If $2 \leq |k| \leq p$ but $k \neq p\epsilon_i$ for any i , then in fact $\zeta^{[k]}$ is in the image of the natural map $J^2 \subseteq \mathcal{P} \rightarrow \mathcal{D}(1)$. Since ϕ^b is of order ≤ 1 , this implies $\phi(\zeta^{[k]}) = 0$. Finally, if $k = p\epsilon_i$, then $\phi(\zeta^{[k]}) = \phi(\zeta_{m_i}^{[p]}) = 0$. \square

Using this criterion along with (A.6), it is easy to show that both $(D_1 + D_2)^p - D_1^p - D_2^p$ and $(aD)^p - a^p D^p$ are PD differential operators of order ≤ 1 . Therefore, for instance, $(aD)^p - a^p D^p = (aD)^{(p)} - a^p D^{(p)}$, which implies $\psi_{\pi^*}(aD) = a^p \psi_{\pi^*} D$, and similarly for the additivity.

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